An introduction to the Universal Coefficient Theorem

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1. Introduction. Given C*-algebras $A$ and $B$ with modest hypotheses, G. G. Kasparov [21] has defined groups $K_{Ki}(A, B)$ ($j \geq 0, 1$) which play a fundamental role in the modern theory of C*-algebras and, particularly, its application in areas related to global analysis and algebraic topology. In this paper we prove a Künneth Theorem which determines the Kasparov groups in terms of the periodic K-theory groups $K_n(B)$ of Karoubi and the Brown-Douglas-Fillmore groups $K^*(A)$, and we establish a Universal Coefficient Theorem (UCT) of the form

$$\text{0 Ext}(K_n(A), K_n(B)) \xrightarrow{\text{K}} K^*(A, B) \xrightarrow{\text{Hom}(K_n(A), K_n(B))} \text{0}$$

which determines the Kasparov groups in terms of K-theory. These short exact sequences are split, unnaturally. When $B = \mathbb{C}$ (the complex numbers) we obtain [with a new, perhaps simpler, proof] the UCT of L. G. Brown [5] for the Brown-Douglas-Fillmore groups $K^*(A) \xrightarrow{\text{K}} K^*(A, \mathbb{C})$.

A great deal of the power of Kasparov's theory comes from the existence of a "Kasparov intersection product" with good functorial properties, generalizing all of the usual products (cup, cap, slant, etc.) in topological K-theory. For reasons to be explained later, our UCT also determines this product structure. In particular, we determine the structure of the graded ring $K^*(A, A)$.

Our results should be useful in several situations where the $KK$-groups are encountered. These include the classification of extensions of C*-algebras, index theory on foliated manifolds (as in [10]), and index theory for elliptic operators with "coefficients" in a C*-algebra (as in [25] and [30], 3B). For instance, a family of elliptic pseudodifferential operators over a compact manifold $M$ with parameter space $Y$ defines an element of $KK_{\lor}(C(M), C^0(Y))$; hence the computation of this group is of some interest. In Section 8 we discuss some applications to the "algebraic topology" of C*-algebras. The computation of the graded ring $KK_{\lor}(A, A)$ and its graded module $KK_{\lor}(A, C^0(A), A)$ in a few basic cases makes it possible for us to determine all of the "homology operations" and "admissible multiplications" for mod $p$ K-theory of C*-algebras.

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[RS] p. 434

Difficulties, we henceforth assume that $A$ is separable nuclear and that $B$ has a countable approximate unit throughout the paper unless stated otherwise.

[RS] p. 439

**Universal Coefficient Theorem (UCT)** 1.17. Let $A \in \mathcal{N}$. Then there is a short exact sequence

$$0 \rightarrow \text{Ext}^1_\mathbb{Z}(K_\ast(A), K_\ast(B)) \xrightarrow{\delta} KK_\ast(A, B) \xrightarrow{\gamma} \text{Hom}(K_\ast(A), K_\ast(B)) \rightarrow 0$$

which is natural in each variable. The map $\gamma$ has degree 0 and the map $\delta$ has degree 1.
Brown’s UCT

\[0 \rightarrow \text{Ext}_\mathbb{Z}^1(K^0(X), \mathbb{Z}) \rightarrow \text{Ext}(C(X)) \rightarrow \text{Hom}(K^1(X), \mathbb{Z}) \rightarrow 0\]
Here are our principal theorems. Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear $C^*$-algebras which contains the separable Type I $C^*$-algebras and is closed under strong Morita equivalence (by [7], this is the same as stable isomorphism), inductive limits, extensions, and crossed products by $\mathbb{R}$ and by $\mathbb{Z}$. We may also require that if $J$ is an ideal in $A$ and $J$ and $A$ are in $\mathcal{N}$ then so is $A/J$, and if $A$ and $A/J$ are in $\mathcal{N}$ then so is $J$. As pointed out by Skandalis [39],

Universal Coefficient Theorem (UCT) 1.17. Let $A \in \mathcal{N}$. Then there is a short exact sequence

$$0 \to \text{Ext}^1_\mathbb{Z}(K_*(A), K_*(B)) \to KK_*(A, B) \to \text{Hom}(K_*(A), K_*(B)) \to 0$$

which is natural in each variable. The map $\gamma$ has degree 0 and the map $\delta$ has degree 1.
We consider

\[ \gamma(A, B) : KK_*(A, B) \to \text{Hom}(K_*(A), K_*(B)) \]

- If \( K_*(B) \) is injective, \( I \triangleleft A \), and two out of \( \gamma(I, B) \), \( \gamma(A, B) \), \( \gamma(A/I, B) \) are isomorphisms, so is the last.
- If \( K_*(B) \) is injective, \( A = \lim_{\to} A_i \), and all \( \gamma(A_i, B) \) are isomorphisms, so is \( \gamma(A, B) \).
- If \( K_*(B) \) is injective then \( \gamma(C_0(X), B) \) is an isomorphism.
- If \( K_*(B) \) is injective and \( A \) is type I then \( \gamma(A, B) \) is an isomorphism.
[RS] proof, last steps

- If $K_*(B)$ is injective, and $A \in \mathcal{N}$, then $\gamma(A, B)$ is an isomorphism.

- For any $\sigma$-unital $B$ there is $\varphi : B \to D$ with $K_*(D)$ injective and $\varphi_* : K_*(B) \to K_*(D)$ injective.
It is quite possible that the UCT (1.17) holds for completely arbitrary separable C*-algebras $A$. (This is assuming $B$ has a countable approximate unit.

An interesting open problem is to determine whether the UCT might in fact hold for all separable C*-algebras. The argument of Corollary 7.5 shows that this is equivalent to the question: is every separable C*-algebra KK-equivalent to a commutative C*-algebra?* To obtain still another formulation, let $A$ be any

*(Added May, 1986) Recent work of G. Skandalis now shows this is not the case, though this may be true for nuclear separable C*-algebras.
Proposition [RS]
If $A \in \mathcal{N}$, then $A$ is $KK$-equivalent to some $C_0(X)$.

Theorem [Skandalis]
The following are equivalent for a separable $A$ (non necessarily nuclear!)

1. The UCT holds for $A$ and any $B$
2. $A$ is $KK$-equivalent to some $C_0(X)$
3. If $K_*(B) = 0$, then $KK(A, B) = 0$

and there is a non-nuclear $A$ for which they are false.
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Theorem [Elliott]

For $A$ and $B$ AT-algebras of real rank zero, we have

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \iff (K_*(A), K_*(A)_+) \cong (K_*(B), K_*(B)_+)$$
The UMCT [Dadarlat-Loring]

For $A \in \mathcal{N}$ we have

$$\text{Pext}(K_*(A), K_*(B)) \quad \Rightarrow \quad \text{Hom}(K_*(A), K_*(B))$$

$$\text{Ext}(K_*(A), K_*(B)) \quad \Rightarrow \quad \text{Hom}_\Lambda(K(A), K(B))$$
Theorem [Dadarlat-Loring]

For $A$ and $B$ AD-algebras of real rank zero, we have

$$A \otimes K \cong B \otimes K \iff (K(A), K(A)_+) \cong (K(B), K(B)_+)$$
Theorem [Kirchberg-Phillips]

Suppose $A$ and $B$ are simple, separable, nuclear, purely infinite $C^*$-algebras. If $A$ and $B$ are $KK$-equivalent, then $A \otimes K \cong B \otimes K$.

Theorem [Kirchberg-Phillips]

Suppose $A$ and $B$ are simple, separable, nuclear, purely infinite $C^*$-algebras with $A, B \in \mathcal{N}$. Then

$$A \otimes K \cong B \otimes K \iff K_*(A) \cong K_*(B)$$
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Theorem [Tikuisis-White-Winter]
If $A$ is separable, nuclear and satisfies the UCT, then any amenable trace on $A$ is quasidiagonal.

Theorem [Dadarlat]
If $A$ is separable, exact, residually finite-dimensional and satisfies the UCT, then $A$ is AF-embeddable.
New classes

A satisfies the UCT when

- \( A = C^*(G) \) for certain amenable groupoids \( G \) (Tu)
- \( A \) may be locally approximated with UCT subalgebras (Dadarlat)
- \( A = C^\pi_\pi(G) \) for a nilpotent group (Eckhart-Gillaspy)
- \( A \) has a Cartan subalgebra (Barlak-Li)

Localizations

The UCT holds for all nuclear \( C^* \)-algebras if \( \mathcal{O}_2 \) is unique with \( K_*(\mathcal{O}_2) = 0 \) among the purely infinite, nuclear \( C^* \)-algebras [Kirchberg].
### Theorem [Kirchberg]

Suppose $A$ and $B$ are separable, nuclear, purely infinite $C^*$-algebras with

$$\text{Prim}(A) \cong X \cong \text{Prim}(B)$$

If $A$ and $B$ are $KK(X)$-equivalent, then $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

### Theorem [Meyer-Nest, Bentmann-Köhler]

Suppose $A$ and $B$ are separable, nuclear, purely infinite $C^*$-algebras with $A, B \in \mathcal{N}$ and

$$\text{Prim}(A) \cong X \cong \text{Prim}(B)$$

with $X$ a finite accordion space. Then

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \iff K_*(X, A) \cong K_*(X, B)$$