EXERCISES FOR WEDNESDAY

As usual, throughout $G$ will be finite group, $k$ a field of positive characteristic $p$ dividing $|G|$, and $R := H^*(G, k)$, the cohomology ring of $G$.

(1) Given $r \in \mathbb{R}^{>1}$, prove that for any $kG$-module $M$, the $kG$-module $M/\langle r \rangle$ is $V(r)$-torsion and that

$$\text{supp}_G(M/\langle r \rangle) = \text{supp}_G M \cap V(r).$$

Extend this to ideals in $R$.

(2) Use the preceding exercise to prove that any closed subset of Proj $R$ is the support of some finite dimensional $kG$-module.

(3) For any $\mathfrak{p}$ in Proj $R$ and $kG$-modules $M, N$, the natural map between cohomology and Tate cohomology induces an isomorphism of $R_\mathfrak{p}$-modules

$$\text{Ext}^*_G(M, N)_\mathfrak{p} \rightarrow \widehat{\text{Ext}}^*_G(M, N)_\mathfrak{p}.$$

This is another reason why it does matter that one works with $H^*(G, k)$ action on $\text{StMod} kG$ rather than that of the Tate cohomology ring.

(4) Let $E = (\mathbb{Z}/3)^2$, the elementary abelian 3-group of rank 2, and $k$ a field of characteristic three. Its cohomology is of the form $\Lambda_k(x_1, x_2) \otimes_k k[y_1, y_2]$, with $|x_i| = 1$ and $|y_i| = 2$.

Describe $\Omega^{-1}k$ and $\Omega^{-2}k$, and the classes $x_i$ and $y_i$. Use this and the recipe outlined in today’s first lecture to describe $\Gamma_{(0)}k$.

(5) Let $k$ be a field of characteristic two, and

$$R := k[x, y, z]/(x^2 + xy + y^2, x^2y + xy^2),$$

where $|x| = 1 = |y|$ and $|z| = 4$. This is the cohomology of $Q_8$; check this. Verify that Proj $R$ consists of a single point, namely, $(x, y)$.

(6) Let $k$ be a field of odd characteristic and $R$ the ring $k[x, y, z]/(x^2 + y^2 + z^2)$. Prove that this is a domain and find a generic point for $(0)$.