Recall

$$
TC^{-}(R) = THH(R)^{hT}
$$

$$
TC^{*}_{*}(F_p) = Z_p[x, t]/(xt - p)
$$

with $\deg(t) = -2$ and $\deg(x) = 2$.

Norm maps: Let $G$ be a cpt. Lie group and $X$ a spectrum with (left) $G$-action in the sense of higher algebra. The norm map

$$(X \otimes D_{a})^{hG} \rightarrow X^{hG}$$

is the universal example of a map to $(-)^{hG}$ from a functor $\text{Fun}(BG, Sp) \rightarrow Sp$ that preserves all colimits. Can calculate dualizing obj. to be

$$D_{a} = \text{Ad}(G),$$

the one-pt. compactification of the adjoint representation.

Define Tate spectrum to be the cofiber of norm map,

$$\xymatrix{ (X \otimes D_G) \ar[r] & X \ar[r] & X.}$$

Prop The functor $(-)^{t_G}$ admits a unique lax symmetric monoidal structure s.t.

$$( - )^{h_G} \rightarrow ( - )^{t_G}$$

is lax symmetric monoidal.  \[ E_x^* \pi_* ((HZ)^{t_G} C_\ell) = H^{-*}(C_\ell, \mathbb{Z}) \\
= \mathbb{F}_p [t^\pm 1], \quad \deg (t) = -2. \]

Prop For $k$ a commutative ring and $A$ a $k$-algebra (ordinary rings),

$$HP_* (A/k) = HC_*^-(A/k) [t^{-1}]$$

$$= \pi_* (HH(A/k)_{tT})$$

(For $C^{tT} \simeq C^{hT} [t^{-1}]$.)
Rank more generally, for \( X \in \text{Fun}(\text{BST, ModMU}) \), one has

\[
X^t_t = X^{h_t} [e^{-1}],
\]

where \( e \) is the Euler class of \( T_t \)-repr. \( e(1) \).

**Def** \( TP(R) = T^{HHH}(R)^t_t \)

**Prop** \( TP^*(\mathbb{F}_p) = \mathbb{Z}_p [x, t^{\pm 1}] / (xt - p) = \mathbb{Z}_p [t^{\pm 1}] \).

**Proof** There is a spectral seq.

\[
E_\infty^{i,j} = H^{-i}(\text{TT}, T^{HH}_j(\mathbb{F}_p))
\]

\[
\Rightarrow TP_{i+j}(\mathbb{F}_p)
\]

\[\text{Diagram with nodes at } (t^2, t, 1, t^{-1}, t^{-2}) \text{ and arrows indicating relationships.}\]
Tate diagonals (see N-S):

There is a unique map

\[ X \xrightarrow{\Delta p} (X \otimes_{\mathcal{E}} EX)^{tG} \quad p \]

which is natural in \( X \) and compatible w. Lax symm. monoidal structures.

Exercise For an abelian group, \( A \)

\[ A \twoheadrightarrow \hat{H}^0(C_p, A) \quad p \]

a \( p \)-class of \( \text{ae} \)

is additive and exhibits the target as mod \( p \) reduction of the source, i.e., induces

\[ A/\mathfrak{p}A \twoheadrightarrow \hat{H}^0(C_p, A^{\mathfrak{p}}) \]

For any \( A \)-ring \( R \), we get a Frobenius map

\[ \text{TTHH}(R) \xrightarrow{\Delta p} \text{TTHH}(R)^{tG} \]

making the diagram
\[ R \longrightarrow \operatorname{THH}(R) \]
\[ \xrightarrow{\Delta p} \quad \xrightarrow{\Phi_p} \]
\[ (R^\otimes p)^{tG_p} \longrightarrow \operatorname{THH}(R)^{tG_p} \]

commute. It is given by
\[ R^\otimes 3 \cong R^\otimes 2 \cong R \]
\[ \xrightarrow{\Delta p} \quad \xrightarrow{\Delta p} \quad \xrightarrow{\Delta p} \]
\[ (R^\otimes 3)^{tG_p} \cong (R^\otimes 2)^{tG_p} \cong (R^\otimes p)^{tG_p} \]

Since \( \Phi_p \) is equivariant w.r.t.
\[ \xrightarrow{\Pi \Pi / G_p} \]
\[ \xrightarrow{\Pi / G_p} \]
\[ (\otimes p, G_p) \]

**Lemma:** If \( X \) is bounded below, then the canonical map
\[ \xrightarrow{X^{\Pi / G_p}} \]
\[ (X^{tG_p}) \]

is a \( p \)-completion.

**Proof:** After \( p \)-completion,
\[ \begin{align*}
\lim_{n \to \infty} X \times \mathbb{Q}_p^n & \cong \lim_{n \to \infty} (X \times \mathbb{Q}_p)^n_{\mathbb{Q}_p} \\
& \cong (X \times \mathbb{Q}_p)^{\mathbb{Q}_p} \\
\end{align*} \]

and use Tate lemma to show
\[ X \times \mathbb{Q}_p^n \cong (X \times \mathbb{Q}_p)^{\mathbb{Q}_p} \]

Both claims uses that Tate constr. preserves Postnikov towers to reduce to \( X = \oplus \mathbb{H}_{\mathbb{Q}_p} \).

Thus we can interpret \( \mathbb{C}_{\mathbb{Q}_p}^{\mathbb{Q}_p} \) as
\[ T^\infty H(R) \cong (T^\infty H(R)^{\mathbb{Q}_p})^\wedge. \]

Can combine all prime numbers \( p \) to get a map
\[ T^\infty H(R) \cong \bigoplus_p \left( T^\infty H(R)^{\mathbb{Q}_p} \right)^\wedge \]

\[ \left( T^\infty H(R)^{\mathbb{Q}_p} \right)^\wedge \]

to pro-finite completion.
Prop: The ring map \[ T \mathbb{C} (\mathbb{F}_p) \xrightarrow{\varphi_p} T \mathbb{P}(\mathbb{F}_p) \]

\[ \mathbb{Z}_p[x, t]/(xt - p) \xrightarrow{\varphi_p} \mathbb{Z}_p[t^\pm 1]/(xt - p) \]

is given by

\[ \varphi_p(x) = \lambda \cdot t^{-1} \]

\[ \varphi_p(t) = \lambda^{-1} \cdot pt \]

for some \( \lambda \in \mathbb{Z}_p^* \).

Remark will show next time that, in fact, \( A = 1 \). Note, in part, that the map is isomorphism in non-negative degrees and has image \( p^n \mathbb{Z}_p \subset \mathbb{Z}_p \) in degree \(-2n\).

Proof. Since ring map, given by

\[ \varphi_p(x) = \lambda \cdot t^{-1}, \quad \varphi_p(t) = \mu \cdot t \]

with \( \lambda, \mu \in \mathbb{Z}_p^* \). One of these must be a unit, since \( xt = p \). Suppose \( \mu \in \mathbb{Z}_p^* \). This implies that \( \varphi_p \) is a unit in
non-pes. degrees. Now

\[ TTHH(F_p) \xrightarrow{\phi} TTHH(F_p) \xrightarrow{t \pi} (HF_p) \]

\[ \xrightarrow{\phi} \quad \xrightarrow{t \pi} \quad \xrightarrow{t \pi} \]

commutes and is given in deg. -2 by

\[ \mathbb{Z}_p, \text{t } \longrightarrow \mathbb{Z}_p, \text{t } \longrightarrow \mathbb{F}_p, \text{t } \]

\[ \xrightarrow{\phi} \quad \xrightarrow{\phi} \]

so assumption was wrong. So be \( \mathbb{Z}_p^* \), and \( x \cdot t = p \) shows that \( \mu = \mathbb{Z}_p^{1-\phi} \), since \( \phi \) mult. 11

Def: If \( R \) is a ring or a comm. ring spectrum, then

\[ TC(R) \longrightarrow TC^+(R) \xrightarrow{\eta} TP(R) \]

\[ \eta \]

B the equalizer.

Thus this definition agrees with old definition of (Goodwillie’s "integral") \( TC(R) \) based on
Bökstedt's definition of $THH(\mathbb{F}_p)$. 

Prop. $TC_* (\mathbb{F}_p) = \mathbb{Z}_p [\epsilon]/(\epsilon^2)$ with $\text{deg}(\epsilon) = -1$.

In part, $TC(\mathbb{F}_p)$ is a module over $\mathbb{Z}_p = \mathbb{Z}_p \cdot TC(\mathbb{F}_p)$, hence an Eilenberg-MacLane $\mathbb{F}_p$.

Proof. Have code.

$\begin{array}{ccl}
\text{can, } \varphi & : & TC_* (\mathbb{F}_p) \longrightarrow TP_* (\mathbb{F}_p) \\
\mathbb{Z}_p [x, t]/(xt-p) & \longrightarrow & \mathbb{Z}_p [x, t^{+1}]/(xt-p)
\end{array}$

to be

\[
\begin{align*}
\text{can}(t) &= t, \quad \text{can}(x) = pt^{-1} \\
\varphi_p(t) &= px^{-1}t, \quad \varphi_p(x) = xt^{-1},
\end{align*}
\]

so $\text{can} - \varphi_p$ is an isom., except in degree zero.