Define $\text{THH}(R) = \text{HH}(R/\mathbb{S})$:

\[ R \otimes_{\mathbb{S}} R \otimes_{\mathbb{S}} R \otimes_{\mathbb{S}} R = R \]

Ex: $\text{THH}(\mathbb{S}) = \mathbb{S}$.

There is a functor

\[ D(\mathbb{Z}) \longrightarrow \text{Sp} \]

which is lax symmetric monoidal:

\[ (hc) \otimes_{\mathbb{S}} (hd) \longrightarrow H(C\otimes_{\mathbb{Z}} D) \]

By abstract nonsense, we get a map for $R \in \text{Alg}(D(\mathbb{Z}))$

\[ \text{THH}(HR) \longrightarrow H(\text{HH}(R/\mathbb{Z})) \]

Prop: For an ordinary ring, the map

\[ \text{THH}_j(R) \longrightarrow \text{HH}_j(R/2) \]

is an isomorphism, for $j \leq 2$.

Pf: Both sides only depend on $\mathbb{S}$.
\[ \mathbb{I}_{\leq 2} \mathbb{H} \mathbb{R} \cong \mathbb{I}_{\leq 2} \mathbb{H} \mathbb{R} \]

\[ \mathbb{I}_{\leq 1} (\mathbb{H} \mathbb{R} \otimes \mathbb{H} \mathbb{R}) \cong \mathbb{I}_{\leq 1} \mathbb{H} \mathbb{R} \otimes \mathbb{I}_{\leq 1} \mathbb{H} \mathbb{R} \]

\[ \mathbb{I}_{\leq 0} (\mathbb{H} \mathbb{R} \otimes \mathbb{H} \mathbb{Z} \otimes \mathbb{H} \mathbb{Z}) \cong \mathbb{I}_{\leq 0} \mathbb{H} \mathbb{R} \otimes \mathbb{I}_{\leq 0} \mathbb{H} \mathbb{Z} \otimes \mathbb{I}_{\leq 0} \mathbb{H} \mathbb{Z} \]

As a result, we get that

\[ \text{THH}_2(\mathbb{H}_{\mathbb{F}_p}) \cong \text{HH}_2(\mathbb{F}_p / \mathbb{Z}) , \]

so get \( \text{THH}_2(\mathbb{H}_{\mathbb{F}_p}) = \mathbb{F}_p - \times \).

**Thm (Bökstedt)** As a graded \( \mathbb{F}_p \)-algebra,

\[ \text{THH}_*(\mathbb{H}_{\mathbb{F}_p}) = \mathbb{F}_p [x] \]

So denominators disappear!

An alternative proof of Bökstedt's theorem (which uses the same ingredients, in the end) is based, for \( p = 2 \), on Mahowald's thm.

therefor the 2-fold loop map

\[ \Sigma^2 S^3 \longrightarrow BO \]

its Thom spectrum \( MF \) is \( HiF_2 \). The Thom spectrum functor
preserves colimits and is symmetric monoidal. It follows that

\[ \text{THH} (M_f) \cong M (B^{cyc} (\Omega^2 S^3 \to \Sigma)) \]

\[ \cong M (L \Omega S^3 \to \Sigma) \]

\[ \cong M (\Omega S^3 \times \Omega^2 S^3 \to \Sigma) \]

\[ \cong \Omega S^3 \otimes M(f) \cong \Omega S^3 \otimes \text{HIF}_2 \]

as A\_\infty-rings. It is easy from some spectral sequence to see that

\[ H_\ast (L \Omega S^3, \text{IF}_2) = \pi_\ast (\Omega S^3 \otimes \text{HIF}_2) \]

\[ = \text{IF}_2 [x]. \]

The analogue of Mahowald's theorem for p odd was proved by Hopf. The "linearization" map

\[ \text{THH} \ast (\text{IF}_p) \to \text{THH} \ast (\text{IF}_p / \mathbb{Z}) \]

\[ \text{IF}_p [x] \to \text{IF}_p [x] \]

\[ \text{IF}_p [x] \to \text{IF}_p [x] \]
$\beta$ is the unique $F_p$-algebra map sending $x$ to $x$. It is zero in degrees $\geq 2p$.

Recall the $A = k[z]/(z^2)$ - action on $\text{HH}(R/k)$.

Thus the $\infty$-category of DG-$k$-modules over $A$ is equivalent to the $\infty$-category of chain complexes with a $T = \mathfrak{gl}(1)$ - action:

$$\text{Fun}(BT, D(k)).$$

\textbf{Pf (Sketch)} $\text{Fun}(BT, D(k))$ is equivalent to DG-$k$-modules over $C^* (T, k)$ so they follow from formality equivalence

$$C^* (T, k) \simeq A.$$ 

Exercise: Prove this formality statement.

Prop: For every $R \in \text{Alg}(C)$ there is a canonical $T$-action on $B^G E_k (R)$.
Proof (Sketch) Recall that $B^{cy}(R)$ is defined to be the comp. 

\[ N \Delta^\text{op} \xrightarrow{\text{cut}} \text{Ass}_{\text{act}} \xrightarrow{\otimes} \mathcal{C}. \]

The functor "cut" extends to Connes' cyclic category $\Lambda$, so we get 

\[ N \Lambda^\text{op} \xrightarrow{\text{cut}} \text{Ass}_{\text{act}} \xrightarrow{\otimes} \mathcal{C}, \]

and for every cyclic object 

\[ N \Lambda^\text{op} \xrightarrow{\otimes} \mathcal{C}, \]

the colimit of the restriction 

\[ N \Delta^\text{op} \xrightarrow{\otimes} N \Lambda^\text{op} \xrightarrow{\otimes} \mathcal{C} \]

admits a canonical $\mathbb{II}$-action. This is based on the fact that $|N \Lambda^\text{op}| \approx \mathbb{BT}$. 

So we have a canonical $\mathbb{II}$-act. on $\text{HH}(R/k)$, also for $k = S$. In these terms, 

\[ \text{HC}^-(R/k) = \pi_*(\text{HH}(R/k)^{\mathbb{II}}). \]
and
\[ \varphi_*(R/k) = \varphi_*(\text{HH}(R/k))^{\mathbb{T}T} \].

These definitions also make sense for \( k = S \).

Definition: For \( R \in \text{Alg}(S) \), define
\[ TC^-(R) = \text{THH}(R)^{\mathbb{H}T} \]
\[ TP(R) = \text{THH}(R)^{\mathbb{T}T} \].

Warning: \( TC^-(R) \) is not (yet) standard notation, and \( TC(R) \) is not \( \text{THH}(R)^{\mathbb{H}T} \).

Proposition: As graded rings,
\[ TC_*(\mathbb{F}_p) = \mathbb{Z}_p[x, t^t] / (xt - p) \]
\[ TP_*(\mathbb{F}_p) = \mathbb{Z}_p[x, t^{t+1}] / (xt - p) \].

Proof: Consider homotopy fixed points, spectral sequence (resp. Tate spectral sequence), which in homological Serre grading, i.e.,
\[ E_{ij}^0 \rightarrow E_{i-r, j+r-1}^r \].
looks as follows:

\[ \begin{array}{cccc}
& t^4 & t^3 & t^2 & t \\
1 & 1 & 1 & 1 & 1 \\
x & x & x & x & x \\
x^2 & x^2 & x^2 & x^2 & x^2 \\
x^3 & x^3 & x^3 & x^3 & x^3 \\
x^4 & x^4 & x^4 & x^4 & x^4 \\
\end{array} \]

Every \( x = E_p \) and \( t \) in even total degree, so \( E^\infty = E^2 \).
To solve extension problems, it suffices to show that

\[ x + t \in E^\infty 
\]

represents \( p \in TC_0^-(\mathbb{F}_p) \).
(We know from spectral sequence that the class in \( TC_0^- (\mathbb{F}_p) \)
represented by \( x + t \) is nonzero.)
But this is true in

\[ HC_x^- (\mathbb{F}_p) = \mathbb{Z}_p \langle x \rangle [t] / (xt - p), \]

and since \( x \) and \( t \) are in the image of \( TC_0^- (\mathbb{F}_p) \to HC_x^- (\mathbb{F}_p) \),
the relation holds there, too, \( \square \)