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Outline:
- Review $\text{HH}$ and $\text{THH}$
- $\text{TP}$ and $\text{TC}^-$
- $\text{TC}$
- Cyclic $\text{torsion}$ spectra

**Def:** If $R$ is an assoc. ring, then
\[
\text{HH}_*(R) = \pi_* (\text{HH}(R)), \quad \text{where}
\]
\[
\text{HH}(R) = (\cdots \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R})
\]

Ex: $\text{HH}_*(\mathbb{Q}) = \mathbb{Q}$

For $R$ comm. $\text{HH}_j(R) \otimes_{\mathbb{R}} \mathbb{Q} = \mathbb{Q}, \quad j \leq 1$.

**Lemma:** If $R$ is commutative, then $\text{HH}_*(R)$ is an anticommutative graded ring.

**Pf:** $\text{HH}(R)$ is CDGA, since assoc. w. simplicial commutative ring.
Thm (HKR) If \( k \) is a comm. ring and \( R \) a smooth \( k \)-algebra, then

\[
\Omega^*_R(k) \cong HH_*(R/k).
\]

Ex

\[
HH_*(k[x_1, \ldots, x_n]/k) = k[x_1, \ldots, x_n] \otimes_k \Lambda_k \{dx_1, \ldots, dx_n\}.
\]

What is \( HH_*(\mathbb{F}_p/\mathbb{Z}) \)? Literally, get \( \mathbb{F}_p \) in degree 0, but \( \mathbb{F}_p/\mathbb{Z} \) is not flat, so want to derive tensor products. Equivalently, replace \( R/k \) by \( \mathcal{E} \mathcal{D} \mathcal{G} \mathcal{A} \) \( R/k \) that is degreewise flat \( R/k \), and define

\[
HH(R/k) := HH(\hat{R}/k).
\]

Prop

\[
HH_*(\mathbb{F}_p/\mathbb{Z}) = \Gamma_{\mathbb{F}_p} \{x\} , \quad \text{deg} x = 2.
\]

Pt In general,

\[
HH(R/k) = R \otimes_k R^{op},
\]

In case at hand, use

\[
\mathbb{F}_p = \mathbb{Z}[\varepsilon]/(\varepsilon^2), \quad d\varepsilon = p.
\]
to get
\[ \mathbb{F}_p \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong \mathbb{F}_p [E_1/(\varepsilon^2)] = A, \]
with zero differential. Resolve
\[ \Gamma_A : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}_p \]
with \( d \times [\varepsilon] = \varepsilon [\varepsilon - 1] \varepsilon. \) The lemma follows. \( \Box \)

Connes (1982): For \( R/k \) associative, there is extra structure on the complex \( HH(R/k) \), namely
\[ HH(R/k)[n] \xrightarrow{B} HH(R/k)[n+1] \]
\( s.t. \ dB = B\delta \) and \( B^2 = 0. \) It is defined by
\[ B(r_0 \otimes \ldots \otimes r_n) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\sigma} (r_{\sigma(0)} \otimes \ldots \otimes r_{\sigma(n)}) \]
with \( \sigma \in \mathfrak{S}_n \). Equivalently, this equips \( HH(R/k) \) with a str. of DG-module over
\[ A = k[E_1/(y^2)] = H_* (U(1), k) \]
with \( \|y\| = 1. \)
The cyclic homology $B$

$HC_*(R/k) = \text{Tor}^A_* (k, HH(R/k))$,

the negative cyclic homology $B$

$HC_*^-(R/k) = \text{Ext}^A_* (k, HH(R/k))$,

and periodic cyclic homology $B$

$HP_*^-(R/k) = HC_*^-(R/k)[\frac{1}{t}]$

with $t \in \text{Ext}_A^2(k, k)$.

Here $\text{Ext}^A_*(k, k) = k[t]$, $|t| = 2$.

Exercise: If $q < k$, and $R/k$ sm., then

$HC_*^j(R/k) = \mathbb{Z}_{R/k} \otimes \bigoplus_{i=0}^j H^{i+2i}_d(R/k)$

$HP_*^j(R/k) = H^*_d(R/k)(lt))$.

Prop: $HC_*^-(\mathbb{F}_p/\mathbb{Z}) = \mathbb{Z}_p<x>[[t]]$

$HP_*^-(\mathbb{F}_p/\mathbb{Z}) = \mathbb{Z}_p<x>(lt)) / (xt-p)$

$= (\mathbb{Z}_p<y>/(y-p))^+ [t^{-1}]$

where $y = xt$. 


Note that \( \text{HP}_* (\mathbb{F}_p / \mathbb{Z}) \) has \( p \)-torsion, for example
\[
( y^{[p]} - \frac{p}{p!} ) \cdot p = 0.
\]

A more abstract perspective:
We use the symmetric monoidal \( \infty \)-categories
\[
\left( D(\mathbb{Z}), \odot^\mathbb{Z} \right), \quad \left( \text{Sp}, \otimes^\mathbb{S} \right).
\]

Def. We define an (ordinary) category \( \text{Ass}^\otimes \) (act = active) whose objects are finite sets and whose morphisms are maps of finite sets together with linear orders on preimages of elements.

This has a symmetric monoidal structure given by disjoint union, and in this symmetric monoidal category, the object \( \mathbb{1} \) has an associative algebra structure.

Ex. Check that for every symmetric monoidal category \( \mathcal{C} \), evaluation
induces an equivalence categories

\[ \text{Fun}^\otimes (\text{Ass}_{\text{act}}, \mathcal{C}) \sim \text{Alg}(k). \]

**Def:** If \( \mathcal{C} \) is a symm. monoid.al
\( \infty \)-category, then an associative
algebra in \( \mathcal{C} \) is a symm. mon.
functor

\[ \text{NAss}^\otimes \mathcal{C} \]

**Ex:** If \( R \) is a flat DGA \( /k \), then
there is a symm. mon. functor

\[ \text{NAss}^\otimes \mathcal{D}(k) \]

which takes \( S \) to \( R \otimes S \).

There is a functor

\[ \Delta^\otimes \mathcal{P} \mathcal{D} \mathcal{Ass}_{\text{act}} \]

which sends a finite linearly
ordered set \( S \) to the set

\[ \text{Cut}(S) = \{ S = S_0 \sqcup S_1 | S_0 < S, \} / \sim \]

where \( \emptyset \sqcup S \sim S \sqcup \emptyset \). (This
is the cyclic bar construction for the alg. 2,1 in $\text{Ass}_{\text{act}} \otimes$.

Def let $R$ be an assoc. alg. in a symm. monoidal $\infty$-category $\mathcal{E}$. The cyclic bar construction of $R$ is the composite functor

$$\Delta^{op} \to \text{Ass}_{\text{act}} \otimes \mathcal{E}.$$ 

Can consider its realization

$$B^{gec}(R) = \text{colim} B^{gec}(R)[\cdot-1],$$

where $B^{gec}(R)[\cdot-1]$ is above comp.

Prop If $R/k$ is a flat DGA considered as an algebra in $D(k)$, then

$$B^{ygec}(R/k) \simeq \text{HH}(R/k).$$