Morrow I
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Fix prime number $p$ for whole course.

Introduction to $p$-adic Hodge theory:
Goals: Indicate goals and objects of $p$-adic Hodge theory:

- Introduce main objects of BMS I:
  "Integral $p$-adic Hodge theory"

- Mention rel. to THTH in BMS II:
  "Integral $p$-adic Hodge th. and THTH"

Introduction:
$K$: complete discrete valuation field of char. 0 with perf. residue field $k$ of char. $p$.
$O_K$: ring of integers
$X$: proper sm. Sch. / $O_K$.

\[ \xymatrix{ & X \ar[dl] \ar[dr] & \quad \text{special fiber} \quad \text{generic fiber} \quad \text{geom. gen. fiber} \\
X_k \ar[rr] & & X \times_k X \times_k \overline{k} } \]
In d’adic coh., the proper and sm.
basechange theorem gives isom.
\[ H^n_{et}(X_k, \mathbb{Z}_l) \cong H^n_{et}(X_F, \mathbb{Z}_l) \]
for any prime \( l \neq p \) (compatible
with Galois actions), False if \( l = p \)!

Grothendieck (70s): Should replace
\[ H^n_{et}(X_k, \mathbb{Z}_p) \]
by "crystalline coh."
\[ H^n_{rys}(X_k / W(k)) \] and compare
\[ H^n_{rys}(X_k / W(k)) \leftrightarrow H^n_{et}(X_F, \mathbb{Z}_p) . \]

Note: LHS related to diff. forms;
RHTS replacement for Betti coh.
So Grothendieck asking for alg.
analogue of \( (M \text{ sw. mfld. } \mathcal{C}) \)
\[ H^n_{dR}(M / \mathcal{C}) \cong H^n_{B}(M, \mathcal{C}) \]
\[ \uparrow \sim \]
\[ H^n_{dR}(M / \mathbb{R}) \otimes \mathbb{C} \]
\[ H^n_{B}(M, \mathbb{Z}) \otimes \mathbb{C} / \mathbb{R} \]
Precise answer proposed by Fontaine
after inverting \( p \)
There is a natural isomorphism

\[ H^n_{\text{cris}}(X_k/W(k)) \otimes_{\mathbb{W}(k)} \mathbb{B}_{\text{cris}} \cong H_{\text{ét}}^n(X_E, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\text{cris}} \]
compliant with Galois action etc.

(\mathbb{B}_{\text{cris}} is Fontaine's crystalline p-adic period ring; a big \(\mathbb{Z}_p\)-algebra containing \(\mathbb{W}(k)\)).

Why care about this? Easy cor. (if you understand \(\mathbb{B}_{\text{cris}}\)) is:

\[ (H^n_{\text{ét}}(X_E, \mathbb{Z}_p) \otimes \mathbb{B}_{\text{cris}})_{\text{Gal}(\mathbb{F}/\mathbb{K})} \cong H^n_{\text{cris}}(X_k/W(k)) \otimes_{\mathbb{W}(k)} \mathbb{F}_p^{1/p} \]

i.e. coh. of \(X_E\) + Galois action gives coh. of \(X_k\).

Problem: Since \(\frac{1}{p} \notin \mathbb{B}_{\text{cris}}\), we only see rational cohomology.

Integral p-adic Hodge theory: Do this without inverting \(p\).
Main idea of BMS I: We define a new coh. th. $H^n_{\text{dR}}(X)$ that interpolates

$$H^q_{\text{et}}(X_K/W(k)), \ H^q_{\text{et}}(X_K, \mathbb{Z}_p),$$

and $H^q_{\text{dR}}(X/\mathcal{O}_K^\infty)$.

This coh. th. is the hypercoh. of a certain ca. of presheaves in the Zariski topology on $X \times \mathcal{O}_K^\infty$:

$$X \times \mathcal{O}_K^\infty \rightarrow \text{Spec}(\mathcal{R}) \ (\mathcal{R}/\mathcal{O}_K^\infty \text{ sm.})$$

$$\rightarrow A\Omega\hat{\mathcal{R}}$$

where $A\Omega\hat{\mathcal{R}}$ is a certain ca. of $\text{Ainf}$-modules, depending only on the $p$-adic completion $\hat{\mathcal{R}}$, and where $\text{Ainf}$ is Fontaine's infinitesimal period ring.

$$A\text{inf} = \lim \ W_r(\mathcal{O}_K^\infty).$$

Main goal of course: Define this $E_\infty$-algebra $A\Omega\hat{\mathcal{R}}$. 
Rel. to TTHH: For $R/0^k$ sm., we prove:

**Thm (BMS II)** There exists a natural descending filtration on $\text{TP}(R)$ such that

$$\text{gr}^i \text{TP}(R)_p \cong A_{\Omega R}^{[2i]}$$

compatible with $A_{\text{inf}}$ - module str. via $\tau_0 (\text{TP}(0^k)_p) = A_{\text{inf}}$.

**Outline** of course:
- Décalage functor.
- DR cx. and Cartier isom.
- DRW cx. and Cartier isom.
- Constr. of $A_{\Omega R}$.
- Appl. to $p$-adic Hodge th.
- Rel. to TTHH.
- TTHH of smooth alg. 1 k.

Décalage functor: Modifying torsion. (Deligne, Berthelot-Ogus, Mazur)

Fix ring $A$, non-zero divisor for A.
Def: If $C$ is a cx. of $A$-mod. s.t.
(1) $C^n = 0$ for $n < 0$
(2) $C^n$ is $f$-tors. free for $n \geq 0$,
then define subcomplex
$\gamma f C \subset C$
by
$(\gamma f C)^n = \{ x \in f^n C^n \mid dx \in f^{n+1} C^{n+1} \}$. \\

Note: $H^n(\gamma f C) \to H^n(C)$ has kernel and cokernel killed by $f^n$.

Lemma: The map $C^n \to (\gamma f C)^n$ given by $x \mapsto f^n x$ induces $\lambda$.

\[ H^n(C)/H^n(C)[f] \sim \to H^n(\gamma f C), \]

where $(\cdot)[f]$ is submodule annihilated by $f$.

Cor: If $C \to C'$ is a qis, then so is $\gamma f C \to \gamma f C'$.

Cor: For every cx. $D$ of $A$-mod. s.t.
(1') $H^n(D) = 0$ for $n < 0$
(2') $H^0(D)$ is $f$-tors. free
then we may define derived functor

\[ H \mathcal{D} := \mathcal{F} \mathcal{C} \]

where \( \mathcal{C} \) satisfies (1) and (2),
and \( \mathcal{C} \subset \mathcal{D} \).