Applications to p-adic Hodge theory:

Thin (Globalization) For every smooth (formal) O- scheme X, there is a natural complex of sheaves \( A_{\mathcal{O}X/0} \) in the \( \varphi \) _- topology s.t. for every open \( \text{Spec}(R) \subset X \),

\[
A_{\mathcal{O}X/0}(\text{Spec}(R)) = A_{\mathcal{O}X/0}. 
\]

Similarly have \( W_{\mathcal{O}X/1} \), \( \mathcal{O}_{X/1} \).

Pf Use different approach to \( A_{\mathcal{O}X/0} \), namely Scholze's pro-étale coh. of rigid analytic spaces (instead of Galois cohomology).

Def For \( X/0 \) smooth proper,

\[
\mathcal{R} \Gamma (X) = \mathcal{R} \Gamma (X, A_{\mathcal{O}X/0}). 
\]

Its cohomology \( H^n_{\text{Ainf}}(X) \) are \( \text{Ainf} \)-modules of finite type.
Comparison theorem:

\[ R_{A^{\infty}}(X) \otimes A^{\infty} \cong R_{\text{proj}}(X, \mathbb{Z} \times \mathbb{Q}) \]

\[ R_{A^{\infty}}(X) \otimes W(k) \cong R_{\text{proj}}(X \otimes W(k)) \]

\[ R_{A^{\infty}}(X)[\frac{1}{\mu}] \cong R_{\text{et}}(X \otimes A^{\infty}[\frac{1}{\mu}]) \]

\[ \cong R_{\text{et}}(X \otimes \mathbb{Z}_p) \otimes A^{\infty}[\frac{1}{\lambda}] \]

Back to THTH: From now, complete everything at \( p \).

Thus \( \text{THTH}_*(0, \mathbb{Z}_p) = 0 \otimes \mathbb{Z}_l \), \( |\mathbb{Z}_l| = 2 \).

Pf Recall \( A^{\infty} = W(0^b) \), and

\[ A^{\infty}/pA^{\infty} = 0^b \]

\[ A^{\infty}/\mathbb{F}_p A^{\infty} = 0 \]

Since \( 0^b \) is perfect

\[ \mathbb{L} A^{\infty}/\mathbb{Z}_p / p \cong \mathbb{L} 0^b/\mathbb{F}_p \cong 0 \]
So \( \text{Ext} /\mathbb{Z}_p \cong 0 \). Hence

\[ HH_* (0 /\mathbb{Z}_p) \cong HH_* (0 /\text{Ann}_0 \mathbb{Z}_p) = 0^* \times \mathbb{Z}, \ 1 \times 1 = 2. \]

So enough to show that

\[ \text{THH}_* (0, \mathbb{Z}_p) \rightarrow \text{HH}_* (0, \mathbb{Z}_p) \]

is injective w. image in \( \text{HH}_{2m} \), for \( n = 2m \). To do this, use Pirashvili - Waldhausen sp. seq.

\[ E_{i,j}^2 = \text{HH}_i (0, \mathbb{Z}_p, M_j) \]

\[ = \text{THH}_{i,j} (0, \mathbb{Z}_p) \]

\[ M_j = \begin{cases} 0, & \text{if } j = 0 \\ \mathbb{Z}, & \text{if } j = 2m \\ 0 / m \mathbb{Z}, & \text{if } j = 2m - 1. \end{cases} \]

(This works for any ring.) Since \( 0 /\mathbb{Z}_p \) is flat \( M_j = 0 \) for \( j > 0 \) even, and since primes \( t \neq p \) are invertible in \( \mathbb{O} \),

\[ 0 / m \mathbb{O} = 0 / p^j \mathbb{O} \]

for \( j > 0 \).
The $E^2$-term looks as follows

\[ j_{=2p-1} \frac{HH_0(0)}{p} \circ \frac{HH_2(0)}{p} \circ \frac{HH_2(0)}{p} \]

\[ j_{=0} \frac{HH_0(0/2p)}{p} \circ \frac{HH_2(0/2p)}{p} \circ \frac{HH_4(0/2p)}{p} \]

and the map

\[ \text{THH}_n(0, Z_p) \rightarrow \text{HH}_n(0/2p) \]

is the edge-homomorphism to the baseline. So

\[ \frac{HH_{2m}(0/2p)}{\text{im}(\text{THH}_{2m}(0, Z_p))} \]

\[ = \frac{E_{2m,0}}{E_{2m,0}} \]

has a filtration with filtration quotients killed by

\[ p, 2p, \ldots, L \frac{m}{p} \]

the product of which is $p^{m}$ (exercise). So $E_{2m,0} / E_{2m,0}$ is
hitted by \( m! \). Conversely,

\[ \text{THH}_* \left( 0, \mathbb{Z}_p \right) \otimes \mathbb{L} \xrightarrow{\sim} \text{THH}_* \left( \mathbb{L} \right) \]

which by Böckenholt's Theorem is equal to \( \mathbb{L}[x_1] \). This can happen only if

\[ E_{2m,0}^0 = 0, \ x^m, \]

and since

\[ E_{2m,0}^2 = 0, \ \frac{x^m}{m!}, \]

we conclude that \( E_{2m,0}^2 / E_{2m,0}^0 \) precisely is \( 0 / m! \cdot 0 \). This, in turn implies that \( E_{i,j}^0 = 0 \) for \( j > 0 \), so the map from \( \text{THH}_* \left( 0, \mathbb{Z}_p \right) \) to \( \text{HTH}_* \left( 0, \mathbb{Z}_p \right) \) is injective.

Thm If \( R / 0 \) is smooth, then

\[ \delta_{R / 0} \otimes \text{THH}_* \left( 0 \right) \xrightarrow{\sim} \delta_{R / 0} \otimes \text{THH}_* \left( \mathbb{L} \right) \]

Pf Using Pirashvili-Waldhausen spectral sequence to reduce to same assertion for \( \text{THH}_* \left( - / 0 \right) \), and here it is easy.
Then let $R$ be the $p$-completion of a smooth $\mathbb{Q}$-algebra. The following spectra have natural filtrations with graded pieces as indicated:

$$\text{gr}^i \text{THH}(R)$$
$$\simeq \left( \mathbb{Z} \left[ \mathcal{L}_{R/\mathbb{Q}} \right] \right)_{2i}$$

$$\text{gr}^i \left( \text{THH}(R)^{Cp^n} \right)$$
$$\simeq \left( \mathbb{Z} \left[ \mathfrak{w}_{r+1} \mathcal{L}_{R/\mathbb{Q}} \right] \right)_{2i}$$

$$\text{gr}^i \left( \text{THH}(R)^{tP} \right)$$
$$\simeq \mathbb{A} \Omega_{R/\mathbb{Q}} \left[ 2i \right]$$

Everywhere, $\text{THH}(R)$ should really be $\text{THH}(R, \mathbb{Z}_p) = \text{THH}(R)^{\wedge}_p$.  

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Pirashvili-Waldhhausen sp. seq.:

$$E_{i,j}^2 = \text{HH}_{i,j}^{\mathbb{Q}}(A/\mathbb{Z}, \text{THH}_j(\mathbb{Z}, A))$$

$$\Rightarrow \text{THH}_{i+j}^\wedge(A).$$

Also $\text{THH}_{2m-1}(\mathbb{Z}) \simeq \mathbb{Z}/m\mathbb{Z}$ and htpy. grps. in pos. even degree vanish.