Goal: $\text{THH}$ of smooth algebra over $\mathbb{F}_p$ (or more generally over perfect field).

Thm (Bökstedt) $\text{THH}_* (\mathbb{F}_p) = \mathbb{F}_p [x]$.

Now let $R$ be a smooth $\mathbb{F}_p$-algebra. Then (this does not use smoothness)

$$\text{THH}_1 (R) \xleftarrow{\sim} \text{HH}_1 (R/\mathbb{F}_p) \xleftarrow{\sim} \Sigma^1 \mathbb{F}_p / \mathbb{F}_p.$$ 

Using product structure, get

$$\Sigma^* \mathbb{F}_p / \mathbb{F}_p \longrightarrow \text{THH}_* (R),$$

compatible with differentials.

Thm (Hesselholt) The map of (differentially graded) rings

$$\Sigma^* \mathbb{F}_p / \mathbb{F}_p \otimes \text{THH}_* (\mathbb{F}_p) \longrightarrow \text{THH}_* (R)$$

is an isomorphism, for $R/\mathbb{F}_p$ smooth.

So $\text{THH}_n (R) = \bigoplus \Sigma^{n-2i} \mathbb{F}_p / \mathbb{F}_p$. First new result is to upgrade this, to spectrum level result.
Thin (B-M-S II) There is a natural, decreasing filtration of \( \text{THH}(R) \) (in \( \text{Fun}(\mathbb{N}(\mathbb{N}), \mathbb{S}p) \)) such that

\[
\text{gr } \text{THH}(R) \cong \left( \mathbb{Z} \text{^i} \Sigma_{R/\mathbb{F}_p} \right) [2i]
\]

where \( \mathbb{Z} \text{^i} \Sigma_{R/\mathbb{F}_p} \) is the ax.

\[
R \xrightarrow{\Omega_{R/\mathbb{F}_p}} \Sigma_{R/\mathbb{F}_p} \xrightarrow{\Omega^i_{R/\mathbb{F}_p}} \ker(d) \xrightarrow{\sim} \]

Taking \( \pi_*^* \), this recovers calc. above, using Cartier isom.

\[
\Sigma^i_{R/\mathbb{F}_p} \xrightarrow{\sim} H^i \left( \Sigma_{R/\mathbb{F}_p} \right).
\]

Goal: Sketch proof and state generalizations.

Tools from homotopical algebra:

- For any alg. \( A/\mathbb{F}_p \), pick simp. resolution \( PE_1 \rightarrow A \) by degree-wise free \( \mathbb{F}_p \)-alg., i.e. cofibrant res. in \( \text{Alg}(\mathbb{F}_p) \) \( \text{Set} \), and consider the simplicial cochain complex

\[
\Omega_{PE_1/\mathbb{F}_p}
\]
Define:

- The $i$'th row is the "$i$'th power of the cotangent $\mathfrak{c}x"$, $\mathbb{L} i^i \mathfrak{A}/\mathcal{E}_p$

- The $\oplus$ totalization of the diagram is "derived DR cohomology $\mathfrak{c}x"$, $\mathbb{d}R^i \mathfrak{A}/\mathcal{E}_p$

- Define increasing filtration

$$F^j \mathbb{d}R^i \mathfrak{A}/\mathcal{E}_p \subset \mathbb{d}R^i \mathfrak{A}/\mathcal{E}_p$$

by applying $i \leq j$ to each column.
Obs. 1:

\[ \text{gr}_j \mathfrak{d} \mathcal{R}A/\mathbb{F}_p = H^j_{\text{dR}}(\mathcal{P}[j])[-j] \approx \Omega^j_{\mathcal{P}[j]/\mathbb{F}_p} [-j] = L^j_{A/\mathbb{F}_p} \]

Obs. 2: Give

\[ \text{THH}(A) \overset{\sim}{\leftarrow} |\text{THH}(\mathcal{P}[j])| \]

a filtration with \( n \)th graded piece

\[ \bigoplus_{i=0}^{n-2i} \Omega^i_{\mathcal{P}[j]/\mathbb{F}_p}[i] \]

by taking Postnikov filters in each simplicial degree.

Notation: Say that a functor \( F : \text{Ep-alg} \to \text{D}(\mathbb{Z}), \text{Sp}, \ldots \)

satisfies flat descent, if for any faithfully flat \( A \to B \), the map
$F(A) \rightarrow \dim F(B^{\bullet},F^{-1})$ \\
\[\Delta\]

is a weak equivalence.

By thm. of Bhatt, $L\Lambda^{1}/F$ satisfies flat descent. Combining this with observations, get

Cor For $\mathcal{F}$-algebras,

$THH(-)$ and $\text{Fil}^j\text{d}R^{1}/F$

satisfy flat descent.

Now let $R^{1}/F$ be smooth. Construct new filter on $THH(R)$ as follows. Let

$R^{perf} = \colim (R^{p}\rightarrow R^{p})$.

Since $R^{1}/F$ is smooth,

$R \rightarrow R^{perf}$

is faithfully flat. By cor.,

$THH(R) \rightarrow \dim THH(R^{perf},F^{-1})$ \\
\[\Delta\]
and define descending filtr. by
\[ \text{Fil}^i \text{ThH}(R) = \lim_{\Delta} \Delta \geq 2i \text{ ThH}(R^p[t, \theta_R^{-1}]). \]

**Key lemma.** Let \( S \) be a perfect \( \mathbb{F} \)-algebra, and let \( I \subset S \) be an ideal gen. by a regular sequence. Then
\[ \text{ThH}^i_{2i}(S/I) \cong \text{Fil}^i_1 dR(S/I)/IF_p \]
and the odd homotopy groups vanish.

**Proof.** Classical fact that \( \text{IL}^i(S/I)/IF_p \) is supported in deg. \( i \).
By obs. 2, this shows that the odd degree hitpy groups are zero. By obs. 1,
\[ \text{Fil}^i_1 dR(S/I)/IF_p \]
is supported in degree 0.
Now obs. 1 + obs. 2 imply that LHS and RHS of 20 & in statement have same graded
pieces, so enough to construct a map

\[ \text{Fil}_l \cdot \text{dR}(S/I)/\mathbb{F}_p \rightarrow \text{THH}_{2l}(S/I) \]

inducing this isom. Omitted. \[ \]

\textbf{Proof (of thm. on page 2)}

First recall that

\[ R \text{perf} \otimes_R R \text{perf} \]

has the form \( S/I \) in key lemma. E.g. for \( R = \mathbb{F}_p \llbracket x \rrbracket \),

\[ R \text{perf} \otimes_R R \text{perf} = \mathbb{F}_p \llbracket x/y, y/p^l \rrbracket / (x-y) \]

So by key lemma,

\[ \text{gr}_l \cdot \text{THH}(R) \]

\[ \cong \lim_{\Delta} \text{HH}_{2l}(R \text{perf} \otimes_R R \text{perf}, \mathbb{F}_p) \llbracket 2^l \rrbracket \]

\[ \cong \lim_{\Delta} \text{Fil}_l \cdot \text{dR} R \text{perf} \otimes_R R \text{perf}, \mathbb{F}_p \llbracket 2^l \rrbracket \]

\[ \cong \text{Fil}_l \cdot \text{dR} R/\mathbb{F}_p \llbracket 2^l \rrbracket \]

\[ \cong (2^l \cdot \mathbb{F}_p) \llbracket 2^l \rrbracket \]
where the next to last \( \simeq B \) is just descent for \( \\mathcal{F}^{1,1}_d, R^{-1}F_p' \) and where the last \( \simeq B \) is a consequence of \( R^{-1}F_p' \) being smooth.

Can repeat construction for \( \mathbb{THH}(-) \) replaced by

\[
TR^n(-; p), TP(-), TC^-(\cdot),
\]

\[
TC(-; p), \ldots
\]

to get similar filtrations. Note also that filtration on \( \mathbb{THH}(-) \) is not by \( \Pi \)-spectra.