de Rham complex.

Let $A$ be a commutative ring and $B$ a comm. $A$-algebra, i.e. a ring homom. $A \rightarrow B$. The universal $A$-linear derivation of $B$ into a $B$-module is written

$$B \xrightarrow{d} \Omega^1_{B/A}$$

Explicitly,

$$\Omega^1_{B/A} = \frac{B \otimes A B}{\langle xyez - xeyz + zxye \rangle}$$

with $d(y) = \text{class of } 1 \otimes y$, hence,

$$x \partial y = \text{class of } xey.$$ Note

$$\Omega^1_{B/A} \xrightarrow{\partial} \mathrm{HH}_1(B/A)$$

$$x \partial y \mapsto \text{class of } xey.$$

Ex: If $B = A[T_i; 1 \leq i \leq n]$, then $\Omega^1_{B/A}$ is a free $B$-module on basis $\{dT_i; 1 \leq i \leq n\}$.

In general, if $B/A$ is smooth, then $\Omega^1_{B/A}$ is a $\mathcal{D}_{B/A}$-free $B$-module.
Form de Rham complex as follows

$$\Omega^0_{B/A} = \Lambda^0 B \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \cdots$$

with differential

$$d : \Omega^n_{B/A} \to \Omega^{n+1}_{B/A}$$

given by

$$d(x_0 dx_1 \cdots dx_n) = dx_0 dx_1 \cdots dx_n.$$ 

The map

$$B \to \Omega^1_{B/A}$$

is the universal multiplicative map from $B$ to a CDGA $/ A$. Define de Rham cohomology by

$$H^*_{dR}(B/A) = H^*(\Omega^*_{B/A}).$$

Remark: If $B/C$ is smooth, then

$$H^*_{dR}(B/C)$$

is canonically isomorphic to analytic de Rham cohomology of the complex manifold, defined by $B$. 
In characteristic $p$, interesting things happen:

Thus (Cartier isom., '57) let $R$ be a smooth algebra over a perfect field $k$ of char. $p > 0$. Then the "inverse Cartier operator"

$$
\sigma_{R/k}^{-1} \cong \text{H}^n_{et}(R/k)
$$

$x, y_1, ..., y_n \rightarrow x^p, y_1^{p^{-1}}, ..., y_n^{p^{-1}}$

is well-defined and an isom.

Prove well-defined. To prove isom., do so for $R = k[T_1, ..., T_d]$ by case, and conclude isom. for $R/k$ smooth by étale descent.

Rmk. To make isom., one of $R$-mod., let $R$ act on RHS by $r \cdot w = r^p \cdot w$.

Relation to $L y_p$; (let $k = \mathbb{F}_p$)

Let $\tilde{R}/\mathbb{Z}_p$ be smooth with $\tilde{R}/p\tilde{R} = \mathbb{Z}_p$ and let $\phi: \tilde{R} \rightarrow \tilde{R}$ be a $\mathbb{Z}_p$-algebra map s.t.

$\phi(x) \equiv x^p$ modulo $p\tilde{R}$. Then
$\Phi(dy) = \partial (\tilde{\phi}(y)) = \partial(y^p + px)$

$= py^{p-1}dy + px \in p \Omega \mathbb{E}/\mathbb{Z}_p$

So

$\Phi_n(\tilde{\Omega} \mathbb{E}/\mathbb{Z}_p) < p^n \Omega \mathbb{E}/\mathbb{Z}_p$

and hence,

$\Phi_n(\tilde{\Omega} \mathbb{E}/\mathbb{Z}_p) \subset \eta_p \Omega \mathbb{E}/\mathbb{Z}_p$

This fits in comm. diagram

\[
\begin{array}{ccc}
\Omega \mathbb{E}/\mathbb{Z}_p & \overset{\Phi}{\to} & \eta_p \Omega \mathbb{E}/\mathbb{Z}_p \\
\downarrow & & \downarrow \\
\Omega \mathbb{E}/\mathbb{Z}_p / p & \to & (\eta_p \Omega \mathbb{E}/\mathbb{Z}_p) / p \\
\downarrow & & \downarrow \cong \\
\Omega \mathbb{E}/\mathbb{F}_p & \to & \left[ H_{dR} (\mathbb{E}/\mathbb{F}_p), \text{Bock}_p \right]
\end{array}
\]

and the Cartier isom. is an isom. of complexes as indicated. It follows that

$\Phi_n(\tilde{\Omega} \mathbb{E}/\mathbb{Z}_p) \subset \eta_p \Omega \mathbb{E}/\mathbb{Z}_p$

is a quasi-isomorphism.
Witt vectors: For every ring $A$ and $1 \leq r \leq \infty$, there is a ring

$$W_r(A)$$

such that

1. As a set

$$W_r(A) = A \times \cdots \times A$$

2. The natural "ghost" map

$$W_r(A) \xrightarrow{gh} A^r$$

$$(a_0, \ldots, a_{r-1}) \mapsto (a_0, a_0p, a_0p^2, \ldots)$$

is a ring homomorphism. Here $\mathbb{R}^+H_\mathbb{Z}$ is product ring $A^r$.

**Remark.** If $A$ has no $p$-torsion, then $gh$ is injective, and

$$\mathbb{F}_p A^r \subset gh(W_r(A)) \subset A^r.$$

Ex: $W_r(\mathbb{F}_p) = \mathbb{Z}/p^r \mathbb{Z}$ and

$$W(\mathbb{F}_p) := W_0(\mathbb{F}_p) = \mathbb{Z}/p \mathbb{Z}.$$
There are two ring homomorphisms
\[ W_{r+1}(A) \rightarrow W_r(A) \]
called "restriction" and "Frobenius" and defined by
\[ R(a_0, \ldots, a_r) = (a_0, \ldots, a_{r-1}) \]
\[ F(a_0, \ldots, a_r) = (a_0^p, \ldots, a_r^p) \]
where the formula for F assumes \( p = 0 \) in \( A \). General formulas for \( F \) are more complicated.

Additive "Verschiebung" map
\[ W_r(A) \rightarrow W_{r+1}(A) \]
\[ (a_0, \ldots, a_{r-1}) \rightarrow (0, a_0, \ldots, a_{r-1}) \]
satisfying "projection formula"
\[ a, V(b) = V(F(a), b) \]
and mult. "Teichmüller" map
\[ A \rightarrow W_r(A) \]
\[ a \mapsto (a, 0, \ldots, 0) \]
Ex For every ring $A$, the map

$$
\bigoplus_{l \in \mathbb{N}} W_r(A) \oplus \bigoplus_{l \in \mathbb{N}} W_{r+v_p(l)}(A) \xrightarrow{\cdot [T]} W_r(A[T])
$$

given on $t$'th component in top sum by

$$
\alpha \mapsto \alpha \cdot [T]^r
$$

and on $t$'th component in bottom sum by

$$
\alpha \mapsto v^{-v_p(t)} \left( \alpha \cdot [T]_{r+v_p(l)} \right)
$$

is an isomorphism. Can work out (complicated) formulas for $R$, $F$, $V$, and $F-1r$.

Def. A relative $F$-$V$-procomplex or Witt (pro)complex for $A \to B$ is the data of

$$(W^r, R, F, V, \tau_r) \quad r \geq 1$$

displayed as

$$\begin{align*}
W^3(B) & \to W^3_3 \to W^2_3 \to W^1_3 \to W^0_3 \\
W^3(B) & \to W^3_2 \to W^2_2 \to W^1_2 \to W^0_2 \\
W^3(B) & \to W^3_1 \to W^2_1 \to W^1_1 \to W^0_1
\end{align*}$$

where

- $W^r$ is a CDGA / $W^r(A)$
- $R$ is a map of dg rings
- $F$ is a map of graded rings
- $V$ is a map of graded ab. gps.
- $\tau_r$ ring homom. compatible w. $R$, $F$, and $V$. 

\[ \begin{align*}
\text{s.t.,} \\
RF &= FR, \quad RV = VR, \\
FV &= p \\
Fd \, \lambda_{r+1}(Ibl) &= \lambda_r(Ibl)^{p-1} \, d \lambda_r(Ibl)
\end{align*} \]

where \( b \in B. \]

**Def/cstr. (L-2, H-M, I, H 2015)**

There exists an initial object among relative Witt complexes for \( A \rightarrow B. \) It is written

\[ (W_r - \Omega_{B/A}, R, F, V, \lambda_r)_{r \geq 1} \]

and called the relative DRW complex. \( \forall \]

\[ \text{Rank } W_1 - \Omega_{B/A} = \Omega_{B/A}. \]