Cohomological stabilisation ...

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Toroidal compactifications

How do I construct a (projective) compactification of \( \mathbb{A}^g \) which is

- smooth, or:

- a moduli space for degenerations of abelian varieties?

AMRT 1975:

\[ \Sigma \text{ admissible decomposition of} \]
\[ \text{Sym}^2_{\mathbb{Q}, \text{rat}} (\mathbb{R}^g) \text{ into convex poly. cones} \]
\[ \text{with rat. radical, i.e. kernel is defined over } \mathbb{Q}. \]

Admissibility:

(a) \( \Sigma \) is closed under taking faces of cones or intersections

(b) \( \text{GL}(g, \mathbb{Z}) \otimes \Sigma \) with fin. many orbits.

(c) \( \bigcup_{\sigma \in \Sigma} \text{Sym}^2_{\mathbb{Q}, \text{rat}} (\mathbb{R}^g) \)

\( \Rightarrow \mathbb{A}^g \Sigma \) compactification

(if only have (a) + (b) then get a partial compactif.)
Remark: $\mathcal{A}_g^\Sigma$ is rationally smooth iff all cones in $\Sigma$ are simplicial, i.e. if $\sigma = \mathbb{R}_{>0} \cdot q_1 + \cdots + \mathbb{R}_{>0} \cdot q_n$ with $q_1, \ldots, q_n$ lin. ind. 

The construction of $\mathcal{A}_g^\Sigma$ is based on theory of toric varieties.

$X$, toric variety $\iff \exists T \leq X$ and the $(\mathbb{T}^*)^k$ $T$-action extends to all of $X$.

Idea of construction: max. isotropic flag $U_0 = (0) < U_1 < U_2 < \cdots < U_g$

Fix $U_r \leadsto$ parabolic subgp $P_r = \text{Stab}(U_r) \leq \text{Sp}(2g, \mathbb{Z})$

Generators:

1. $S$ symm. matrix, size $g \times g$

\[
\begin{pmatrix}
I & \text{g-r} & \cdots & \text{g-r} \\
\text{g-r} & I & \cdots & \text{g-r} \\
\vdots & \vdots & \ddots & \vdots \\
\text{g-r} & \text{g-r} & \cdots & I
\end{pmatrix}
\]

Such matrices gen. the (Z-part) of
the centre of the unipotent radical of \( P_r, P_r' \)

\[
p_{r'} \backslash h_g \cong (\text{Symm}^2 (Z^r) \otimes \mathbb{C}^*) \times C^{r(g-r)} \times h_{g-r}
\]

\[
\begin{pmatrix}
I_r & 0 & 0 & 0 \\
0 & A & 0 & B \\
0 & 0 & I_r & 0 \\
0 & C & 0 & D
\end{pmatrix}
\]

where \((A, B) \in \text{Sp}(2g-2r, \mathbb{Z})\)

\[
\begin{pmatrix}
I_r & M & 0 & N \\
0 & I_{g-r} & 0 & N^T \\
0 & 0 & I_r & 0 \\
0 & 0 & 0 & -M^T I_{g-r}
\end{pmatrix}
\]

\(M, N \in M_{r \times (g-r), \mathbb{Z}}\)

Generators of types (2) & (3) generate a subgroup \( \tilde{J} \).

Quotient by \( \tilde{J} \):

\[
\tilde{J} \backslash C^{r(g-r)} \times h_{g-r} = \mathcal{X}^{x_r}
\]

\(r\)th symm. prod. of univ. ab. var. over \( \mathcal{A}_{g-r} \) \( r(r+1)/2 \)

Hence: quotient by \( P_r \) is a \((\mathbb{C}^*)^{r(x_r)}\) bundle over \( \mathcal{X}_{g-r}^{x_r} \)
So: need to construct a compactification of the fibres of the \((\mathbb{C}^*)^{(r+1)/2}\) bundle in a way compatible with the \(P_r/P_r\)-action, and then glue such compactif. together for \(r=0, 1, \ldots, g\).

Consequence: stratification of \(\Sigma\)

\[
\Sigma = \bigcup_{[\sigma] \in \Sigma} \beta_g(\sigma) \quad \text{locally closed stratum}
\]

- Explicit construction:

\[
\begin{align*}
\beta_g(\sigma) & \xrightarrow{\text{finite quotient}} \mathbb{C}^{g-r} \\
\text{Aut}(\sigma) & \xrightarrow{\text{finite quotient}} B_g(\sigma) \rightarrow \mathbb{C}^g
\end{align*}
\]

\(\beta_g(\sigma)\) is rationally smooth of codim. \(\dim_{\mathbb{R}} r\).

- \(\Sigma^g \xrightarrow{\Phi} A_g^S, \quad \Phi(\beta_g(\sigma)) \subseteq A_{g-r}\), where \(r\) is the rank of the quadric forms in \(\sigma\).

Examples of admissible \(\Sigma\) come from the reduction theory of quadratic forms.

**Ex 1:** Perfect cone / 1st Voronoi decomp.

\[\Sigma^P\]

\(\sigma \in \Sigma^P \iff \sigma \in \Sigma \cap R_+ \bar{\xi}_o \cdot \bar{\xi}_o\)

\[\left\{ \bar{\xi}_o \in \mathbb{Z}^g_\text{sym.} / Q(\bar{\xi}_o) < Q(\xi) \forall \xi \in \mathbb{Z}^g \right\}
\]

For some \(Q \in \text{Sym}_2 (\mathbb{R}^g)\).
\[ A_g^p \]

Features:
- irreducible boundary (codim. 1)
- rat. smooth only for \( g \leq 3 \)
- GL-orbits are classified for \( g \leq 7 \)

Ex. 2: 2nd Voronoi compactification \( A_g^v \)

Each \( \sigma \in \Sigma^v \): locus of forms with the same assoc. Delone decomposition of \( \mathbb{R}^g \)

\[
g = 2 \quad \mathbb{R}^2 \quad \begin{array}{c|c|c} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \end{array}
\]

nearest lattice point

\[ \sigma = \langle x_1^2, x_2^2 \rangle \]

\[
\sigma = \langle x_{1,j}^2, x_{2,j}^2, (x_1 - x_2)^2 \rangle
\]

\( A_g^v \leftrightarrow \) moduli space: Alexeev 2002
- rat. smooth for \( g \leq 4 \)
- orbits classified for \( g \leq 5 \)

Rem: \( \Sigma^M = \Sigma^v \cap \Sigma^p \rightarrow A_g^M \) partial compactification of matroidal cones is always rat. smooth.
(Melo-Viviani 2012)
Stability for toroidal comp.

Rem: Product map $P^\Sigma: A^\Sigma_1 \times A^\Sigma_2 \to A^\Sigma$ exists whenever
$\Sigma = \{ \Sigma_0, \Sigma_1, \ldots, \Sigma_g, \ldots \}$ is closed under taking direct sums of cones.
In which case we say $\Sigma$ is additive.

Thm A (Grushevsky-Hulek-T.)

Assume (**) $\dim \sigma > \text{rank } \sigma + 1 \quad \forall_{\sigma \in \Sigma}$

(*) $\Sigma$ is additive

If $\Sigma$ is simplicial then $H^k(A^\Sigma_g)$ stabilises for $k < g$ and is algebraic in this range.

[recall $H^k(-) = H^k(-; \mathbb{Q})$ !]

(*) & (**) \Rightarrow e.g. $\oplus \sigma \ A^\Sigma_g = A^\Sigma_g$

We have $H^2(\mathbb{A}^\Sigma_{\infty}) \cong R \otimes \text{Sym}^* \left( V^\Sigma \right)$ as a free graded algebra

with

$V^\Sigma_k := \bigoplus_{0 \leq \sigma \in \Sigma, \dim_{\text{alg}}(\sigma) = k - \dim_{\text{re}}(\sigma)} \text{Sym}^\sigma \left( \mathbb{Q} - \text{span of } \sigma \right)$

\( \sigma \text{ irreducible part} \Theta \)
Thm B (GHT):

If (**) hold then (Borel-Moore) homology of \( A_{g} \) stabilises w.r.t. the following seq. of Gysin maps

\[
A_{g} \longrightarrow A_{g+1}
\]

transv. (up to finite) cover

\[
\sim \quad H_{\text{top} - k} (A_{g+1}) \longrightarrow H_{\text{top} - k} (A_{g})
\]

"dual" to usual stability maps.

\[\text{for } k < g.\]

and we have an isom. of graded vector spaces

\[
H_{\text{top} - \cdot} (A_{\infty}^{* \Sigma}) \cong R^{\cdot} \otimes \text{Sym}^{\cdot} (V_{\Sigma}^{*})
\]

e.g. \( \Sigma = \Sigma^{p} \).