Nordic Mathematical Team Contest 2016

Problems

Competition time. 28th September, 12.00 CET – 2nd October, 24.00 CET.

Solutions. Solutions must be written in English. Preferred format is a pdf document compiled from a LaTeX source. Scanned solutions written by hand are acceptable, but the jury is at a liberty to deduct points for illegibility.

Submission. Solutions should be submitted to the chairman of the jury at:

`qimh@math.su.se`

Questions. Questions regarding the problems may be directed towards the chairman of the jury. Answers will be posted on the official website.

Score. Each problem is worth 10 points.

Problem 1. Consider a regular 2016-gon D whose inscribed circle has unit radius. Define the function $h\colon D \to \mathbf{R}$ by letting $h(x,y)$ be the shortest distance from (x,y) to the edge of D. Compute the double integral

$$\iint_D h(x,y)\,dx\,dy.$$

Problem 2. What is the value of the infinite product

$$\prod_{p=1}^{\infty}\left(1 - \frac{1}{(4p-2)^2}\right)\ ?$$

Problem 3. A finite abelian group $(G,+)$ with neutral element 0 is said to have the *NMC-property*, if the following holds. There exist subsets $A, B \subseteq G$ with $|A|, |B| \geqslant 2$ such that

$$A \cup B = G, \qquad A \cap B \subseteq \{0\}, \qquad A + A + A \subseteq A \qquad \text{and} \qquad B + B + B \subseteq B.$$

(Here $S + S + S = \{\, s_1 + s_2 + s_3 \mid s_1, s_2, s_3 \in S \,\}$.)

Determine all finite abelian groups with the NMC-property.

Problem 4. Let $f\colon \mathbf{Z} \to \mathbf{R}$. We say that f is periodic with period $a \in \mathbf{Z}^+$ if $f(n+a) = f(n)$ for all $n \in \mathbf{Z}$. We say that $a \in \mathbf{Z}^+$ is the *minimal* period of f if it is the smallest possible period.

(a) Let $f_a, f_b\colon \mathbf{Z} \to \mathbf{R}$ be periodic functions with minimal periods a and b respectively. Show that, if a and b are coprime (so that $\gcd(a,b) = 1$), then $f_a + f_b$ is periodic with minimal period ab. (3 pts)

(b) Construct functions with minimal periods 6 and 10 such that their sum has minimal period 15.

Also, construct functions with minimal periods 18 and 30 such that their sum has minimal period 45. (2 pts)

(c) Let $f_a, f_b\colon \mathbf{Z} \to \mathbf{R}$ be periodic functions with minimal periods a and b respectively (no longer assumed coprime). What minimal periods for $f_a + f_b$ are possible? (5 pts)

Problem 5. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers.

(a) Suppose that $(x_n + x_{n+j^2})_{n=1}^{\infty}$ converges for each $j = 1, 2, \ldots$. Does $(x_n)_{n=1}^{\infty}$ necessarily converge? (3 pts)

(b) Suppose that $(x_n + x_{n+j^3})_{n=1}^{\infty}$ converges for each $j = 1, 2, \ldots$. Does $(x_n)_{n=1}^{\infty}$ necessarily converge? (4 pts)

(c) Suppose that $(x_n + x_{n+2017^j})_{n=1}^{\infty}$ converges for each $j = 1, 2, \ldots$. Does $(x_n)_{n=1}^{\infty}$ necessarily converge? (3 pts)

Problem 6. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function such that

$$g(x) = \lim_{n\to\infty} f^{(n)}(x)$$

exists and is finite for every $x \in \mathbf{R}$. (Here $f^{(n)}$ is the n'th derivative of f.)

Prove or disprove: There is a constant $C \in \mathbf{R}$ such that $g(x) = Ce^x$ for every $x \in \mathbf{R}$.

Problem 7. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a continuous function, satisfying $f(x) \geqslant 0$ everywhere and being unbounded at infinity. Does there necessarily exist a real number α such that the sum

$$\sum_{n=-\infty}^{\infty} f(n\alpha)$$

diverges?

Problem 8. Ali Baba steals treasures from an underground cave in the middle of the desert and then flees from the crime scene. A djinn is sleeping in the City of Brass (somewhere in the desert), but wakes up when he realizes what Ali Baba has done. At the moment Ali Baba leaves the cave, the djinn heads out from the city with the intention of intercepting Ali Baba and punishing him for his crimes.

The djinn knows that Ali Baba will be travelling at a constant speed in a straight line away from the cave. He also knows that all djinns are able to keep a constant speed which is $a > 1$ times faster than Ali Baba's. However, he does not know in which direction Ali Baba is headed.

Devise a trajectory for the djinn to follow so that he will be certain to intercept Ali Baba.

Problem 9.

(a) Let α and β be positive real numbers such that the only integer solution to the equation $\alpha x + \beta y + \alpha\beta z = 0$ is $x = y = z = 0$. Show that every positive rational number is of the form

$$\frac{\lfloor m\alpha \rfloor}{\lfloor n\beta \rfloor}$$

for some positive integers m and n. (7 pts)

(b) Show that the conclusion of part (a) does not hold if α and β are arbitrary positive real numbers. (3 pts)

Problem 10. Define a real-valued function Ψ on the set of real 2×2 matrices by

$$\Psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a^2 + b^2 + c^2 + d^2,$$

and let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & \frac{1763}{42} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \frac{1}{48} & 0 \\ 0 & 48. \end{pmatrix}.$$

Repeatedly tossing a fair coin defines a sequence of 2×2 matrices

$$M_n = \begin{cases} I & \text{if } n = 0; \\ AM_{n-1} & \text{if toss number } n \geqslant 1 \text{ comes up heads;} \\ BM_{n-1} & \text{if toss number } n \geqslant 1 \text{ comes up tails.} \end{cases}$$

For even n, is it more probable than not that

$$\Psi(M_n) \leqslant 2016^n + 1\,?$$