A survey of weak amenability

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References:
1 Definitions

$G$: a locally compact group, 2nd countable. Recall (Leptin):

$G$ is amenable if and only if there exists a sequence of positive definite, compactly supported functions converging to 1 uniformly on compacts subsets.

This can be generalized in the two following ways:
**Definition 1.1.** $G$ has the Haagerup property (or: is a-$(T)$-menable) if there exists a sequence of continuous positive definite functions on $G$, vanishing at infinity, converging to 1 uniformly on compact sets.

**Definition 1.2.** $G$ is weakly amenable (or: has the completely bounded approximation property CBAP) if there exists a sequence $(\phi_n)_{n>0}$ of continuous, compactly supported functions on $G$, converging to 1 uniformly on compact sets, with

$$\sup_n \|\phi_n\|_{M_0A(G)} < +\infty.$$

Here $\|\cdot\|_{M_0A(G)}$ is the completely bounded norm on the space $M_0A(G)$ of completely bounded multipliers of the Fourier algebra $A(G)$. The *Cowling-Haagerup constant* is the best possible $\Lambda$ with $\sup_n \|\phi_n\|_{M_0A(G)} \leq \Lambda$. 
2 Examples of weakly amenable groups

- amenable groups
- closed subgroups of $SO(n, 1)$ (de Cannière-Haagerup 1984) and $SU(n, 1)$ (Cowling 1985) (e.g. free groups)
- Coxeter groups, and more generally groups acting properly on finite-dimensional $CAT(0)$ cubical complexes (Guentner-Higson 2010)

In all those cases $\Lambda(G) = 1$. They also have the Haagerup property.
3 More examples

- $G = Sp(n, 1)$ ($n \geq 2$), with $\Lambda(G) = 2n - 1$; and $G = F_{4(-20)}$, with $\Lambda(G) = 21$ (Cowling-Haagerup 1989)

- hyperbolic groups (Ozawa 2007)

Some of those groups do not have the Haagerup property, because they have Kazhdan’s property (T).
4 Properties of weak amenability

- For $G$ discrete: can be read off from $C^*_r(G)$ and $vN(G)$.
- For discrete groups: invariant under measure equivalence.
- If $\Gamma$ is a lattice in $G$, then $\Lambda(\Gamma) = \Lambda(G)$. (Cowling-Haagerup 1989)

**Theorem 4.1.** If $N$ is a closed, amenable subgroup in a weakly amenable group $G$, then there exists a $N \rtimes G$-invariant state on $L^\infty(N)$ (Ozawa 2010)
Consequence of last result:

**Corollary 4.2.** $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is not weakly amenable (Haagerup 1986)

**Proof:** Assume by contradiction that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is weakly amenable. Let $\phi$ be a $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$-invariant state on $\ell^\infty(\mathbb{Z}^2)$. Decompose $\mathbb{Z}^2$ into $SL_2(\mathbb{Z})$-orbits and observe that each non-trivial orbit is of the form $SL_2(\mathbb{Z})/A$, with $A$ an abelian subgroup of $SL_2(\mathbb{Z})$. By non-amenability of $SL_2(\mathbb{Z})$, the state $\phi$ is zero on any non-trivial orbit, so $\phi$ is evaluation at $(0,0)$, which is of course not $\mathbb{Z}^2$-invariant. □

**Corollary 4.3.** $SL_3(\mathbb{R})$ is not weakly amenable (Haagerup 1986).

Compare with Uffe’s original proof below.
## 5 Permanence properties

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<thead>
<tr>
<th>Property</th>
<th>a-(T)-men</th>
<th>Weak amenability</th>
<th>Remarks</th>
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<tbody>
<tr>
<td>Closed subgroups</td>
<td>Yes</td>
<td>$\Lambda(H) \leq \Lambda(G)$</td>
<td></td>
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<tr>
<td>Products</td>
<td>Yes</td>
<td>$\Lambda(G_1 \times G_2) = \Lambda(G_1)\Lambda(G_2)$</td>
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<tr>
<td>Free products</td>
<td>Yes</td>
<td>$\Lambda(G_1 \ast G_2) = 1$</td>
<td>Only for $\Lambda = 1$</td>
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<tr>
<td>Jolissaint 2001</td>
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<td>OPEN IN GENERAL</td>
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<td>Graph products</td>
<td>Antolin-P</td>
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<td>SAME</td>
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<td>Dreesen 2013</td>
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6 Cowling’s question

Based on experimental evidence, around 1998, M. Cowling conjectured that: the class of Haagerup groups coincides with the class of CMAP groups, i.e. $\Lambda = 1$.

Disproved in 2007:

**Theorem 6.1.** 1. Haagerup property is stable under wreath products (Cornulier-Stalder-V)

2. If $\Lambda \neq \{1\}$ and $\Gamma$ is non-amenable, then $\Lambda \wr \Gamma$ is not weakly amenable (Ozawa-Popa).

So $C_2 \wr F_2$ is an example of a not weakly amenable group with the Haagerup property.

However: Cowling’s conjecture holds for interesting subclasses:

- closed subgroups of $SO(n, 1)$ and $SU(n, 1)$;
- groups acting properly on finite-dimensional $CAT(0)$ cubical complexes (Guentner-Higson).
A finitely generated group $G$ is a \textit{generalized Baumslag-Solitar of rank} $n$ if it admits a co-compact action on some tree, such that vertex and edge stabilizers are isomorphic to $\mathbb{Z}^n$. Such a group admits a canonical homomorphism $hol : G \to GL_n(\mathbb{R})$, the \textit{holonomy representation}.

**Theorem 6.2. (Cornulier-V, 2013)** For a generalized Baumslag-Solitar group $G$ of rank $n$, TFAE:

1. $hol(G)$ is amenable;

2. $G$ has the Haagerup property;

3. $G$ is weakly amenable.

\textit{In that case: $\Lambda(G) = 1$.}
Observations:

- There is no known direct connection between the two properties. In all cases, it is an \textit{a posteriori} observation that a given class of groups satisfy both properties.

- It seems that the discrepancy is related to some lack of finiteness condition.

\textbf{Conjecture 1.} For groups admitting a finite-dimensional $EG$ ($= \text{classifying space of proper actions}$): Haagerup property is equivalent to weak amenability with $\Lambda = 1$
7 A proof of weak amenability

Theorem 7.1. (R. Szwarc, 1991) Let $G$ be a group acting properly on a locally finite tree $T$. Then $\Lambda(G) = 1$.

Examples: free groups, $SL_2(\mathbb{Q}_p)$, ...

Main steps in the proof:

- (Bozejko, Fendler, Gilbert) Let $\phi : G \to \mathbb{C}$ be a continuous function. If there exists continuous, bounded functions $u, v : G \to \mathcal{H}$ such that $\phi(y^{-1}x) = \langle u(x)|v(y) \rangle$ then $\phi \in M_0A(G)$. In particular, for $\pi$ a uniformly bounded representation of $G$, coefficients of $\pi$ belong to $M_0A(G)$.

- (Pytlik-Szwarc 1986, Szwarc 1991, V. 1996) Let $D$ be the open unit disk in $\mathbb{C}$. Fix a base-vertex $v_0 \in T$. There exists an analytic family $(\pi_z)_{z \in D}$ of uniformly bounded representations of $G$ on $\ell^2(T)$ such that:
1. \( \langle \pi_z(g)\delta_{v_0} | \delta_{v_0} \rangle = z^{d(gv_0,v_0)}; \)

2. \( \pi_0 \) is the permutation representation, and \( \pi_t \) is unitary for \( t \in ] -1, 1[; \)

3. \( \sup_{g \in G} \| \pi_z(g) \| \leq \frac{2|1-z^2|}{1-|z|}. \)

- Let \( \gamma_r \) be the circle of radius \( r \) in \( D \), set \( \pi_{\gamma_r} = \int_{\gamma_r} \pi_z |dz| \). Then, for any function \( f \) holomorphic on a neighborhood of \( \gamma_r \), the function \( \phi(g) = \int_{\gamma_r} z^{d(gv_0,v_0)} f(g) \ dz \) is a coefficient of \( \pi_{\gamma_r} \).

- For \( n \in \mathbb{N} \), take \( f(z) = \frac{z^{-(n+1)}}{2\pi i} \). Then \( \chi_n(g) = \frac{1}{2\pi i} \int_{\gamma_r} z^{d(gv_0,v_0)} z^{-(n+1)} \ dz \) is the characteristic function of the sphere \( \{ g \in G : d(gv_0,v_0) = n \} \). Optimizing over \( r \), get \( \| \chi_n \|_{M_0 A(G)} \leq \frac{e}{2}(n + 1) \).
Set $\phi_t(g) = t^{d(gv_0,v_0)}$. For $t \to 1$, it converges to 1 uniformly on compact sets, and $\|\phi_t\|_{M_0 A(G)} = 1$ as $\phi_t$ is positive-definite. BUT: not compactly supported! To fix that, set $\phi_{t,n} = \sum_{k=0}^{n} t^k \chi_k$. Then

$$
\|\phi_t - \phi_{t,n}\|_{M_0 A(G)} = \left\| \sum_{k=n+1}^{\infty} t^k \chi_k \right\|_{M_0 A(G)} \leq \frac{e}{2} \sum_{k=n+1}^{\infty} f^k(k+1)
$$

that goes to 0 for $n \to \infty$. 

8 Recent developments

A weaker notion than weak amenability was introduced by Haagerup and Kraus (1994):
A locally compact group $G$ has the \textit{Approximation Property (AP)} if there is a sequence $(\phi_n)_{n>0}$ in $A(G)$ such that $\phi_n \to 1$ in the $\sigma(M_0A(G), M_0(A(G))_\ast)$-topology, where $M_0(A(G))_\ast$ denotes the natural predual of $M_0A(G)$.

Haagerup and Kraus:

- If $\Gamma$ is a lattice in $G$, $\Gamma$ has AP if and only if $G$ has AP.
- (unlike Haagerup property and weak amenability) AP is stable under extensions (so: $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ has AP).

Establishing a conjecture of Haagerup and Kraus: a simple Lie group of rank at least 2, does not have AP (Lafforgue and de la Salle 2011, Haagerup and de Laat 2013 and 2016).