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Monod III + IV

Ryll-Nardzewski (One person) 1962

Let $K \neq \phi$ be a weakly compact convex set in a Banach space V.

~~Only~~

ASIDE

Weak topology – only closed convex sets are closed.

V reflexive $\Rightarrow \overline{B}$ (the unit ball) is weakly compact

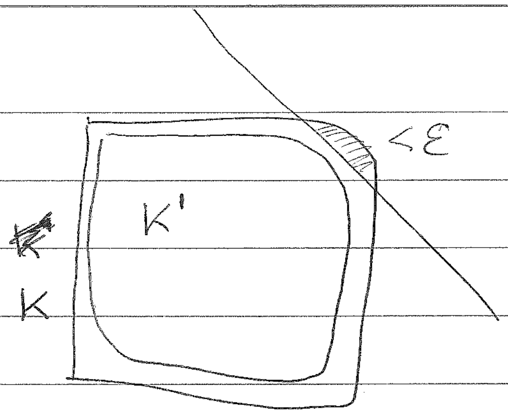
$\exists$ point of K fixed by all isometries (of V preserving K)

Can show existence of Haar measure using this theorem.

Theorem (Lindenstrauss 1963)

Every weakly compact convex $K \subseteq V$ (separable Banach space) is DENTABLE.

i.e. $\forall \varepsilon > 0 \; \exists K' \subseteq K$, K' convex, s.t. $\operatorname{diam}(K \setminus K') < \varepsilon$



Hahn-Banach implies that the denting is via a ~~hat~~ hyperplane.

FACT (Krein-Milman 1940)
Every convex compact set K in any locally convex topological vector space is
$K = \overline{\mathrm{conv}}\{\mathrm{Ext}(K)\}$.

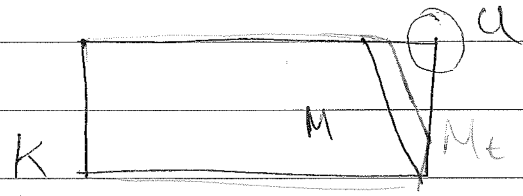
~~Ex~~ Extreme point = not mid point of any two other points.

Proof (Lindenstrauss) uses separability & weakly compact
$L = \mathrm{Ext}(K)$. Baire $\Rightarrow \exists p \in L$ s.t. $B(p,\varepsilon)$ has non-empty interior in L. ($\exists U \subseteq V$, open s.t. $U \cap L \subseteq B(p,\varepsilon)$).
$M = \overline{\mathrm{conv}}(L \setminus U) \subseteq K$
$M_t := \{tm + (1-t)k \mid m \in M, k \in K\}, \quad 0 < t < 1$
$M_t \subseteq K$.
To show for t small, $M_t \neq K$ but
$\mathrm{diam}(K \setminus M_t) < 4\varepsilon$.

If $M_t = K$ then $M = K$, because $\mathrm{Ext}(K) \subseteq M_t$. But this would imply an extreme point would be of the form $tm + (1-t)k \Rightarrow k = m \Rightarrow M = K$.

WLOG $U \not\supset L$ (we can assume $|L| \neq 1$, otherwise the proof is trivial).

$\Rightarrow M \neq K$.



To show $\operatorname{diam}(K \setminus M_t) < 4\varepsilon$. Let $x_1, x_2 \in K \setminus M_t$.

$K = \overline{\mathrm{conv}}\{M \cup \overline{\mathrm{conv}}(L \cap U)\}$

$\Rightarrow x_i = \alpha_i y_i + (1-\alpha_i) z_i$ for $y_i \in M$, $z_i \in \overline{\mathrm{conv}}(L \cap U)$

$(0 < \alpha_i < 1)$. In fact $\alpha_i < t$ (since $x_i \notin M_t$)

$\Rightarrow \|x_i - z_i\| = \|\alpha_i y_i - \alpha_i z_i\| < \alpha_i \operatorname{diam}(K) < t \operatorname{diam} K \leq$
$< \varepsilon$.

$z_i \in B(p, \varepsilon) \Rightarrow \|z_1 - z_2\| < 2\varepsilon$

$\Rightarrow \|x_1 - x_2\| < 4\varepsilon$. $\square$

For RN $G \curvearrowright K$ by isometries, we can assume

1) G is finitely generated $\Rightarrow$ G is countable.

2) V is separable, 3) K is minimal for all hypotheses.

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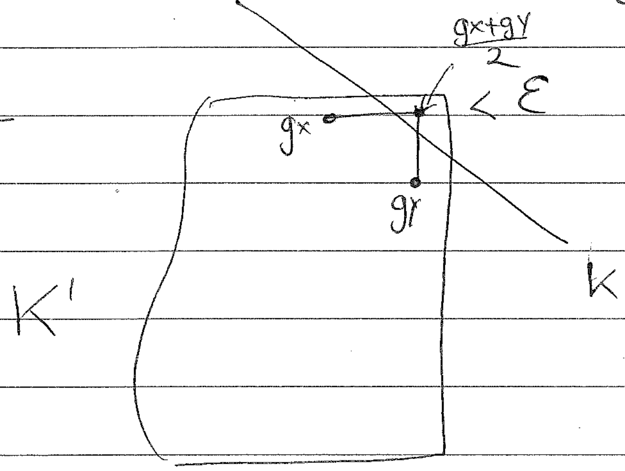
<u>Theorem</u> K is a point!

<u>Proof</u>
Suppose K is not a point. Let $x \neq y$ in K.
Let $0 < \varepsilon < \frac{\|x-y\|}{2}$.

<u>Dent</u>: $\exists\, \emptyset \neq K' \subsetneq K$ convex with $\operatorname{diam}(K \setminus K') < \varepsilon$
Contradiction will be $\emptyset \neq C \subset K'$ convex closed G-invariant.
$C = \overline{\operatorname{conv}}(G \frac{x+y}{2})$, to show $\forall g$: $g \frac{x+y}{2} \in K'$

$\frac{gx+gy}{2}$



But K' is convex. $\square$

<u>FAILURE in L^1</u> $K \subseteq \ell^1(G) = V$
$K = \{\varphi : G \to [0,1] \mid \sum \varphi = 1\}$.
No fixed pt if $|G| = \infty$.

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Roberts 1977

In non LCTVS, $\exists$ convex compact K with $Ext(K) = \emptyset$.

In CAT(0) geometry, there exist examples with $Ext = \emptyset$.

Another proof of RN

$\Gamma \curvearrowright K$, finitely generated

WLOG Γ = f.g. free or $\pi_1($  $)$.

Free group is quotient of $\pi_1($  $)$.

$\Gamma < G$ (lattice) where $G = SL_2(\mathbb{R})$ or $SL_2(\mathbb{Q}_p)$

Define a G-space $\widetilde{K}$ (out of $\Gamma \curvearrowright K$)

s.t G-fixed points in $\widetilde{K} \Rightarrow \Gamma$-fixed points in K.

Idea $K \subseteq V$, $\widetilde{K} = L^2(G/\Gamma, K) \subseteq L^2(G/\Gamma, V)$

or $G \longrightarrow V$, ~~s.t.~~ that are Γ-equivariant

We reduced RN to RN for $G = SL_2(\mathbb{R})$.

$P := \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} < G$ is amenable (b/c ~~meta~~ soluble).

$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow G = P \cup P\omega P$

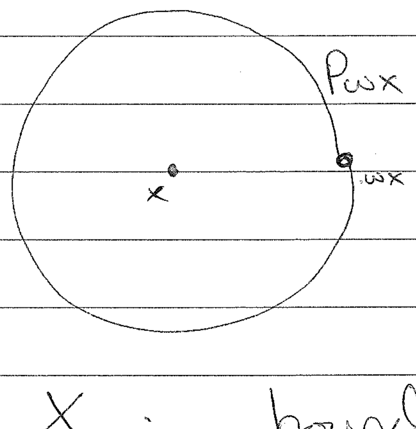
$P\omega P$ is open and dense in G

~~1) G fixes some~~

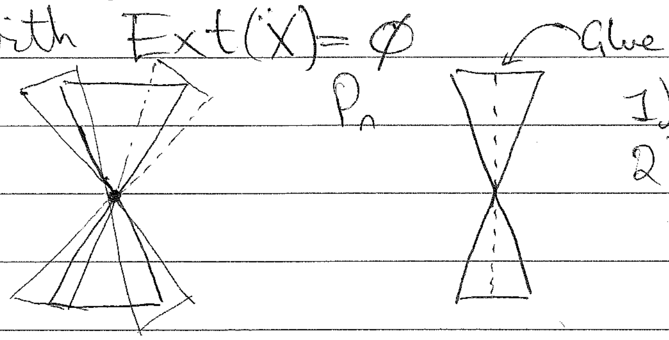
1) P fixes some $x \in K$ (amenability)

2) $Gx = x \cup P\omega x$

$\Rightarrow Gx = \{x\}$. as $P\omega P$ is dense $\square$

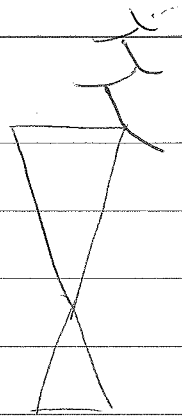


X is bounded 2-dim CAT(0) complete with $\mathrm{Ext}(X) = \emptyset$

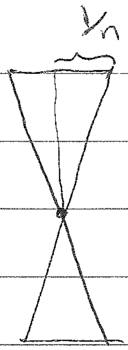


1) Bounded ($\bigcup_{n=1}^{\infty} 2^n P_{n+1}$)

2) Complete



Set P_n



so radius of $\cup 2^n P_{n+1}$ $\leq \frac{\pi}{\sqrt{6}}$.

If a Cauchy sequence visits infinitely many P_n then ~~it~~ from P_n to P_{n+m} the distance is comparable to the angle between the triangles which is the ~~θ~~ harmonic series.