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~~N2~~ Monod I+II

Basic topic

G a group acting on V linear space (Banach) or $K \subseteq V$ convex (compact)

Study G or K or both.

FACT (Banach–Mazur) 1932.

If V is normed, any isometry is affine $(ax+b)$

i.e. $g = \pi(g) + b(g)$, π is linear rep, $b(g) \in V$.

In particular $\pi(gh) = \pi(g)\pi(h)$,
$b(gh) = \pi(g)b(h) + b(g)$.

[$ghv = \pi(gh)v + b(gh) = g(hv) = \pi(g)(\pi(h)v + b(h)) + b(g)$

Proposition: G is finite $\Longleftrightarrow$ every affine G-action has a FP (fixed point).

Proof "$\Rightarrow$" Let $p \in V$ and let $\bar{p} = \frac{1}{|G|} \sum_{g \in G} g.p$

Then $\bar{p}$ is G-fixed. Convex combination so action commutes. ~~with~~

"$\Leftarrow$" $V = \mathbb{R}[G] = \{$linear combinations of elements in $G\}$

$V_0 = \{$elements of sum $= 0\}$, $b(g) = \pi(g)v - v$ for $v = \delta_e$.

$\pi(g)v - v \in V_0$. $b(gh) = \pi(gh)v - v = \underbrace{\pi(gh)v - \pi(g)v}_{\pi(g)\,b(h)} + \underbrace{\pi(g)v - v}_{b(g)}$

Suppose $v_0 \in V_0$ is fixed: $\forall g$, $v_0 = \pi(g)v_0 + b(g)$
$= \pi(g)v_0 + \pi(g)v - v \Rightarrow \pi(g)$ fixes $v_0 + v$
$\Rightarrow v_0 + v$ is constant on G.
G infinite $\Rightarrow v_0 = -v$ impossible $\square$

Remark 1) Will also work for $V = \ell^1(G)$, $V_0 = \ell^1_0(G)$
$= \operatorname{Ker}(\ell^1(G) \to \mathbb{R})$.
convex bounded
2) b is a bounded map $G \to \ell^1_0 \Rightarrow G \curvearrowright K$ without fixe points.

Definition
A group is amenable if it has the FP property for all $K \neq \emptyset$ convex compact (inside a linear topological space)
(continuous affine action has a fixed point).

Example Finite, $\mathbb{Z}$, $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$
$\underset{\underset{\mathbb{Z}}{\Uparrow}}{b(n)} = \delta_{-n} - \delta_0$, $(n \cdot f)(m) = f(m-n) + \delta_{-n} - \delta_0$

Unit ball is not compact in ℓ^1_0 as ℓ^1_0 is not closed

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Remark for Følner: G is amenable $\iff \forall \varepsilon > 0\ \forall S \subseteq G$ finite $\exists F \subseteq G$ finite $\forall s \in S$, $|sF \Delta F| < \varepsilon |F|$.

Følner $\implies$ FP: Let $p \in K$, $p_F := \frac{1}{|F|} \sum_{g \in F} gp \in K$.

$\bar{p}$ = accumulation point of the net $\{p_F\}$.

Let $s \in G$, $sp_F - p_F = \frac{1}{|F|} \left(\sum_{g \in sF \setminus F} gp - \sum_{g \in F \setminus sF} gp \right)$

~~$\in |sF$~~

$$\in \underbrace{\frac{|sF \setminus F|}{|F|}}_{< \varepsilon/2} (SK - K) \longrightarrow 0.$$

2000 year old theorem

If V is Euclidean ($\mathbb{R}^n$, Hilbert) and $K \neq \emptyset$ closed, convex bounded, then any G has the F.P. property. (for isometric action)

Proof Circumcentre!

~~Back to V~~

Back to V = any Banach space $\emptyset \neq K \subseteq V$ bounded

Radius $\rho_V(K) = \inf \{ r > 0 \mid \exists c \in V\ \ K \subseteq B(c, r) \}$.

Chebyshev centre $C_V(K) = \{ c \in V \mid K \subseteq \overline{B(c, \rho)} \}$

WARNING: Often empty. (KONYAGIN 1988)

WARNING: Even for K convex, $C_K(K) = \{c\}$ often $c \notin K$ (Klee 60)

Example $K \subseteq L^1([0,1])$, $K = \{0, 1\}$.

$C_{L^1}(K) = \{0 \leq f \leq 1 \mid \int f = \frac{1}{2}\}$.

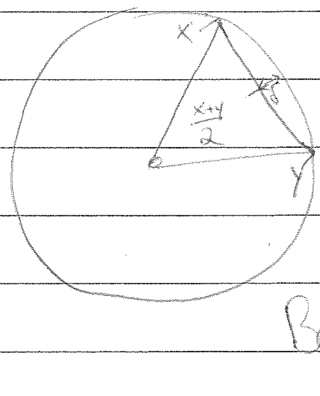
Note $C(K) = \bigcap_{r > \rho} C^r(K)$ for $C^r(K) = \{c \mid K \subseteq \underset{\neq \emptyset}{\overline{B(c,r)}}\}$

$= \bigcap_{x \in K} \overline{B(x,r)}$

$\Rightarrow$ If $\overline{B}$ are compact then $C(K) \neq \emptyset$ (By finite intersection property).

If V is a dual then $\overline{B}$ are weak-$*$ compact.
$\{W \longrightarrow \mathbb{R}$, linear maps$\}$ has norm. Weak-$*$ is topology with pointwise convergence.

Uniqueness holds as soon as V is uniformly convex



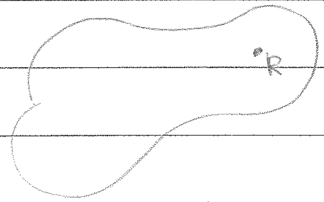
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y \in B(0,1)$

$\|x - y\| > \varepsilon \Rightarrow \|\frac{x+y}{2}\| < 1 - \delta$.

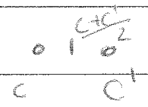
In Euclidean, $\|\frac{x+y}{2}\|^2 = \frac{\|x\|^2 + \|y\|^2}{2} - \frac{\|x-y\|^2}{4}$.

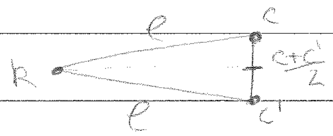
Banach space is CAT(0) $\Leftrightarrow$ It is Hilbert

Uniqueness



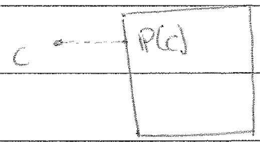
$K \subseteq B(c,\rho)$
$\subseteq B(c',\rho)$.

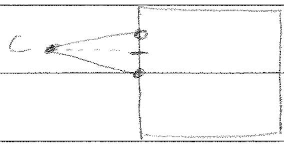




and $d(k, \frac{c+c'}{2}) < \rho - \delta$.

Note Even in the uniformly convex case, K convex, the centre c might be out of K.
But $\exists!\ \mathbb{P}(c) \in K$ (projection of c onto K) such that $d(c, \mathbb{P}(c)) = \inf_{k \in K} d(c,k)$





Uniformly convex spaces $V = L^p, \ell^p \quad 1 < p < \infty$,
U uniformly convex, $L^p([0,1], U)$.