2016.05.06

Lubotzky III + IV

Recall definition of (normalized) expansion. Let X be a simplicial complex of dim. d. For $0 \le i < d$,

$$\varepsilon_i(X) = \min \left\{ \frac{\| d^i(f) \|}{\| [f] \|} \;\middle|\; f \in C^i \setminus B^i \right\}$$

where

$$\| f \| =$$

with

$$w(F) = c(F) \Big/ \binom{d+1}{i+1} |X(d)|$$

$$c(F) = \left| \{ G \in X(d) \mid F \subseteq G \} \right| .$$

Note that w is a probability measure on $X(i)$. Also, if $c(F)$ is constant on $X(i)$, then $w(F) = |X(i)|^{-1}$.

Property testing:

Def $P_n \subset \{0,1\}^n$ is (q, ε)-testable if there exists a randomized algorithm that takes $x \in \{0,1\}^n$ and queries only q bits of it and answers YES if $x \in P_n$ and NO with probability at

$\varepsilon \cdot \overline{\mathrm{dist}}(\alpha, P_n)$

where $\overline{\mathrm{dist}}(\alpha, P_n) = \frac{1}{n}$ Hamming dist (α, P_n). //

Thm Let $W = \mathbb{F}_2^m$ and let $f : W \to \mathbb{F}_2$ be any map. Let V be the $\mathbb{F}_2$-v.s. with basis all such maps; it has dimension 2^m. Is $f : W \to \mathbb{F}_2$ lin.? Algorithm: Pick $x, y \in W$ randomly; check $f(x+y) = f(x) + f(y)$; and answer either YES or NO. This is $(3, 2/9)$-testable. //

This is used for error-correcting codes: To send k bits, encode this as linear map $f : \mathbb{F}_2^m \to \mathbb{F}_2$, for some $m > k$, and use thm. to correct for noise.

Thm Let X be a (finite) simplicial complex, let $n = |X(i)|$; and let $f : X(i) \to \mathbb{F}_2$. Is $f \in B^i$? Test: Pick a random $F \in X(i+1)$ and check $d^i(f)(F) = f(d_{i+1}(F)) = 0$? Answer YES or NO accordingly. This is $(i+2, \varepsilon_i(X))$-testable if and only if X is an i-expander complex.

Ex Let $X=(V,E)$ be a graph, $|V|=n$, $E\subset\binom{[n]}{2}$, $f: V\to\mathbb{F}_2$. Is f const.? If X is a k-regular graph and an expander, then pick $e\in E$ randomly and check if $f(e_+)+f(e_-)=0$. By thm., this is $(2,\varepsilon)$-testable, where ε is the normalized Chieger constant. //

Let $X=(V,E)$, let $U=\{f: E\to\mathbb{F}_2\}$, and call $f\in U$ a sum function if there exists $g: V\to\mathbb{F}_2$ s.t. for every $e\in E$, $f(e)=g(e_+)+g(e_-)$. Let $W\subset U$ be the subspace of sum functions. ($\dim(U)=|E|$, $\dim(W)=|V|$.) Given $f\in U$, is $f\in W$? Do not know how to answer this in general. If X is a complete graph, then a test is given as follows: Pick randomly $i,j,k\in V$ and check if

$$f(i,j)+f(j,k)+f(k,i)=0$$

Answer YES and NO accordingly. This is a $(3,1)$-tester by the Linial-Meshulam theorem from yesterday. ($\varepsilon_1(\Delta_n^{(2)})\geq 1-o_n(1)$).

Ex Let A be a symmetric $m \times m$-matrix with entries ± 1 and with $+1$ on the diagonal. Call A a tensor power if $A = \alpha \otimes \alpha$ with $\alpha \in \{\pm 1\}^m$. Test: Pick randomly i, j, k and check if $A_{ij} A_{jk} A_{ki} = 1$. This is a $(3,1)$-tester by L.-M. //

Def Let X be a simplicial complex purely of dimension d, and let $\varepsilon' > 0$. Say that X is an ε'-geometric (resp. ε'-topological) expander if for every $f: V = X(0) \to \mathbb{R}^d$ and any affine (resp. continuous) extension $\tilde{f}: X \to \mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ which is covered by $\varepsilon' \cdot |X(d)|$ of the d-dimensional cells.

Ex $X = (V, E)$ expander graph, $f: V \to \mathbb{R}$. Pick $z \in \mathbb{R}$ s.t. $|f^{-1}((-\infty, z))| = \frac{1}{2}|V|$. //

Thm (Boros-Furedi) $\Delta_n^{(2)}$ is a $2/9$-geom. expander.

Thm (Barany) There exists $c_d > 0$ s.t. $\Delta_n^{(d)}$ is a c_d-geom. expander.

Thm (Gromov) $\Delta_n^{(d)}$ is a c_d-topological expander for the same c_d as in Barany's theorem.

In fact, Gromov proved:

Thm (Gromov) If a purely d-dim. simpl. cx. X is an ε-coboundary expander, then X is an $\varepsilon' = \varepsilon'(\varepsilon, d)$ topological expander.

Thm (L.-Kaufman-Kazhdan, $d=2$ / S. Evra-T. Kaufman, general d)
For $p \gg d$ prime and X a Ramanujan complex of dim. $(d+1)$ over $\mathbb{F}_p((t))$, the d-skeleton is an $\varepsilon = \varepsilon(d,p)$-topological expander.

Open problem: Are there, in this situation, ε-coboundary expander of bounded degree of dim. $d \geq 2$?

By analogy with ε_i, define

$$\gamma_i = \min \left\{ \frac{\| d^i(f) \|}{\| [f] \|} \;\middle|\; f \in C^i \setminus Z^i \right\}$$

$[f] = f + Z^i$

and call this cocycle expansion. Also, set

$$\eta_i = \min \{ \|\xi\| \mid \xi \in Z^i \setminus B^i \}.$$

Generalization of Gromov's thm.

<u>Thm</u> If $\gamma_i > \varepsilon$ and $\eta_i \geq \varepsilon_1$, then X is an $\varepsilon' = \varepsilon'(\varepsilon, \varepsilon_1)$ - top. exp.

Open problem: Are there LDPC quantum-error-correcting codes?

Namely, $\mathcal{C} = (W_1, W_2)$, $W_1, W_2 \subset \mathbb{F}_2^n$ subspaces s.t. $W_1 \perp W_2$ w.r.t. standard inner product. Define

$$\dim(\mathcal{C}) = \dim(W_1^\perp / W_2)$$

$$k \;\; {}^{\shortparallel} = \dim(W_2^\perp / W_1)$$

$$= n - \dim(W_1) - \dim(W_2)$$

$$\mathrm{dist}(\mathcal{C}) = \min \left\{ w(x) \mid \begin{array}{l} x \in W_1^\perp / W_2 \text{ or} \\ x \in W_2^\perp / W_1 \end{array} \right\}$$

$$d \;\; {}^{\shortparallel}$$

Call $\mathcal{C}$ <u>good</u> if $k/n > \varepsilon$ and $d/n > \varepsilon$, and $\mathcal{C}$ is LDPC if W_1 and W_2

are generated by vectors of bounded weight.

(Sipser – Spielman)

The open problem is whether or not there exists good $\mathcal{C}$ that are LDPC.

Observation: Identifying $C^{i} = C_{i}$,

$W_1 = B_{i}$, $W_2 = B^{i}$

$W_1^{\perp} = Z^{i}$, $W_2^{\perp} = Z_{i}$

$k = \dim H^{i} = \dim H_{i}$.

(L. – Guth)