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Gapped Ground State Phases

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joint work with

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Outline

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- Automorphic equivalence
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What is a quantum ground state phase?

By ground state phase we mean a set of models with qualitatively similar behavior in the ground state(s).

Concretely, this is taken to mean that a g.s. ψ_0 of one model could evolve in finite time to a g.s. ψ_1 of another model in the same phase by some physically acceptable dynamics (one generated by a short-range time-dependent Hamiltonian).

Such dynamics cannot induce or destroy long range order in finite time, and the large-scale entanglement structure remains unchanged.

In the case of gapped phases, the standard definition in the physics literature is that there is a curve of Hamiltonians with finite-range interactions, $H(\lambda), \lambda \in [0,1]$, such that one (or set of) ground state(s) belongs to $H(0)$ and the other to $H(1)$, and such that there is a uniform positive lower bound for the spectral gap above the g.s. for all $\lambda \in [0,1]$ (absence of a quantum phase transition).

- The locality of the interactions is crucial. Automorphisms generated by rapidly decaying interactions are the right middle ground. E.g., such automorphisms satisfy Lieb-Robinson propagation bounds.
- We then explore consequences of this "Automorphic Equivalence". Under the constraint of a symmetry this will lead to an interesting invariant in terms of the symmetry acting on edge states.

(Quasi-local) Automorphic Equivalence

For systems in a finite volume $\Lambda \subset \mathbb{Z}^\nu$, a physically acceptable dynamics is described by a quasi-local unitary V_Λ, solution of the Schrödinger equation:

$$\frac{d}{ds} V_\Lambda(s) = iD_\Lambda(s) V_\Lambda(s), \quad s \in [0,1], \quad V_\Lambda(0) = \mathbb{1},$$

where $D_\Lambda(s)$ is a "Hamiltonian" with short-range interactions:

$$D_\Lambda(s) = \sum_{X \subset \Lambda} \Omega(X, s).$$

When we take the thermodynamic limit to an infinite $\Gamma \subset \mathbb{Z}^\nu$,

$$\lim_{\Lambda \uparrow \Gamma} V_\Lambda(s)^* A V_\Lambda(s) = \alpha_s(A), \quad A \in \mathcal{A}_{\Lambda_0},$$

this dynamics converges to quasi-local automorphisms of the algebra of observables.

The quasi-locality property is expressed as follows: there exists a rapidly decreasing function $F(d)$, such that for observables A supported in a set $X \subset \Gamma$, there exists $A_d \in \mathcal{A}_{X_d}$ such that

$$\|\alpha(A) - A_d\| \leq \|A\| F(d)$$

where $X_d \subset \Gamma$ is all sites of distance $\leq d$ to X.

α is the time evolution for a given unit of time.

For a time-evolution we have something of the form

$$\|\tau_t(A) - A_d\| \leq \|A\| F(d - v|t|)$$

where v is (a bound for) the Lieb-Robinson velocity.

Suppose Φ_0 and Φ_1 are two interactions for two models on lattices Γ.

Each has its set $\mathcal{S}_i$, $i = 0, 1$, of ground states in the thermodynamic limit. I.e., for $\omega \in \mathcal{S}_i$, there exists

$$\psi_{\Lambda_n} \text{ g.s. of } H_{\Lambda_n} = \sum_{X \subset \Lambda_n} \Phi_i(X),$$

for a sequence of $\Lambda_n \in \Gamma$ such that

$$\omega(A) = \lim_{n \to \infty} \langle \psi_{\Lambda_n}, A \psi_{\Lambda_n} \rangle.$$

If the two models are in the same phase, we have a suitably local automorphism α_1 such that

$$\mathcal{S}_1 = \mathcal{S}_0 \circ \alpha_1$$

Fix some lattice of interest, Γ and a sequence $\Lambda_n \uparrow \Gamma$. Let $\Phi_s, 0, \leq s \leq 1$, be a differentiable family of short-range interactions for a quantum spin system on Γ.
Assume that for some $a, M > 0$, the interactions Φ_s satisfy

$$\sup_{x,y\in\Gamma} e^{ad(x,y)} \sum_{\substack{X\subset\Gamma\\ x,y\in X}} \|\Phi_s(X)\| + |X|\|\partial_s\Phi_s(X)\| \leq M.$$

E.g,

$$\Phi_s = \Phi_0 + s\Psi$$

with both Φ_0 and Ψ finite-range and uniformly bounded.
Let $\Lambda_n \subset \Gamma$, $\Lambda_n \to \Gamma$, be a sequence of finite volumes, satisfying suitable regularity conditions and suppose that the spectral gap above the ground state (or a low-energy interval) of

$$H_{\Lambda_n}(s) = \sum_{X\subset\Lambda_n} \Phi_s(X)$$

is uniformly bounded below by $\gamma > 0$.

Theorem (Bachmann, Michalakis, N, Sims (2012))

Under the assumptions of above, there exist automorphisms α_s of the algebra of observables such that $\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$, for $s \in [0, 1]$.
The automorphisms α_s can be constructed as the thermodynamic limit of the s-dependent "time" evolution for an interaction $\Omega(X, s)$, which decays almost exponentially.

Concretely, the action of the quasi-local automophisms α_s on observables is given by

$$\alpha_s(A) = \lim_{n\to\infty} V_n^*(s) A V_n(s)$$

where $V_n(s)$ solves a Schrödinger equation:

$$\frac{d}{ds} V_n(s) = i D_n(s) V_n(s), \quad V_n(0) = \mathbb{1},$$

with $D_n(s) = \sum_{X \subset \Lambda_n} \Omega(X, s)$.

The α_s satisfy a Lieb-Robinson bound of the form

$$\|[\alpha_s(A), B]\| \leq \|A\|\|B\| \min(|X|, |Y|)(e^s - 1)F(d(X, Y)),$$

where $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, $0 < d(X, Y)$ is the distance between X and Y. $F(d)$ can be chosen of the form

$$F(d) = Ce^{-b\frac{d}{(\log d)^2}}.$$

with $b \sim \gamma/v$, where γ and v are bounds for the gap and the Lieb-Robinson velocity of the interactions Φ_s, i.e., $b \sim a\gamma M^{-1}$.

Product Vacua with Boundary States (PVBS)

We consider a quantum 'spin' chain with $n+1$ states at each site that we interpret as n distinguishable particles labeled $i = 1, \ldots, n$, and an empty state denoted by 0.
The Hamiltonian for a chain of L spins is given by

$$H_{[1,L]} = \sum_{x=1}^{L-1} h_{x,x+1},$$

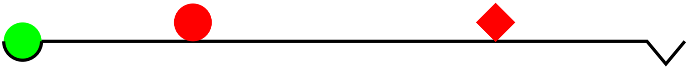
where each $h_{x,x+1}$ is a sum of 'hopping' terms (each normalized to be an orthogonal projection) and projections that penalize particles of the same type to be nearest neighbors.

$$h = \sum_{i=1}^{n} |\hat{\phi}_i\rangle\langle\hat{\phi}_i| + \sum_{1 \leq i \leq j \leq n}^{n} |\hat{\phi}_{ij}\rangle\langle\hat{\phi}_{ij}|,$$

The $\phi_{ij} \in \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ are given by

$$\phi_i = |i, 0\rangle - \lambda_i^{-1}|0, i\rangle \, , \phi_{ij} = |i, j\rangle - \lambda_i^{-1}\lambda_j|j, i\rangle \, , \phi_{ii} = |i, i\rangle$$

for $i = 1, \ldots, n$ and $i \neq j = 1, \ldots, n$.
The parameters satisfy: $\lambda_i > 0$, for $0 \leq i, j \leq n$, and $\lambda_0 = 1$.



There exist $n+1$ $2^n \times 2^n$ matrices $v_0, v_1, \ldots, v_n$, satisfying the following commutation relations:

$$\begin{aligned} v_i v_j &= \lambda_i \lambda_j^{-1} v_j v_i, \quad i \neq j && (1) \\ v_i^2 &= 0, \quad i \neq 0 && (2) \end{aligned}$$

Then, for B an arbitrary $2^n \times 2^n$ matrix,

$$\psi(B) = \sum_{i_1,\ldots,i_L=0}^{n} \mathrm{Tr}(B v_{i_L} \cdots v_{i_1}) |i_1, \ldots, i_L\rangle \qquad (3)$$

is a ground state of the model (MPS vector). In fact, they are all the ground states. E.g., one can pick B such that

$$\psi(B) = \sum_{x=1}^{L} \lambda_i^x |0, \ldots, 0, i, 0, \ldots, 0\rangle$$

If we add the assumption that $\lambda_i \neq 1$, for $i = 1, \ldots, n$, we will have n_L particles having $\lambda_i < 1$ that bind to the left edge, and $n_R = n - n_L$ particles with $\lambda_i > 1$, which, when present, bind to the right edge. The bulk ground state is the vacuum state

$$\Omega = |0, \ldots, 0\rangle \, .$$

All other ground states differ from Ω only near the edges. We can show that the energy of the first excited state is bounded below by a positive constant, independently of the length of the chain. As at most one particle of each type can bind to the edge, any second particle of that type must be in a scattering state. The dispersion relation is

$$\epsilon_i(k) = 1 - \frac{2\lambda_i}{1 + \lambda_i^2} \cos(k) \, .$$

We conjecture that the *exact* gap of the infinite chain is

$$\gamma = \min \left\{ \frac{(1 - \lambda_i)^2}{1 + \lambda_i^2} \, \middle| \, i = 1, \ldots, n \right\} \, .$$

Automorphic equivalence classes of PVBS models

Theorem (Bachmann-N, PRB 2012)

Two PVBS models with $\lambda_i \in (0,1) \cup (1,+\infty)$, $i = 1,\dots n$, belong to the same equivalence class if and only if they have the same n_L and n_R. $l_0 = l_1 = 2^{n_L}$, $r_0 = r_1 = 2^{n_R}$.

Recall that n_L is the number of i such that $\lambda_i \in (0,1)$ and n_R is the number of i such that $\lambda_i \in (1,+\infty)$. l_s and r_s are the dimensions of the ground state spaces of the left and right half-infinite chains.

Conjecture

The dimensions l and r of the ground state spaces of the left and right half-infinite chains are the complete set of invariants for gapped spin chains. Proved for the case $l = r$! (Bachmann-Ogata, CMP 2015).

The AKLT model (Affleck-Kennedy-Lieb-Tasaki, 1987)

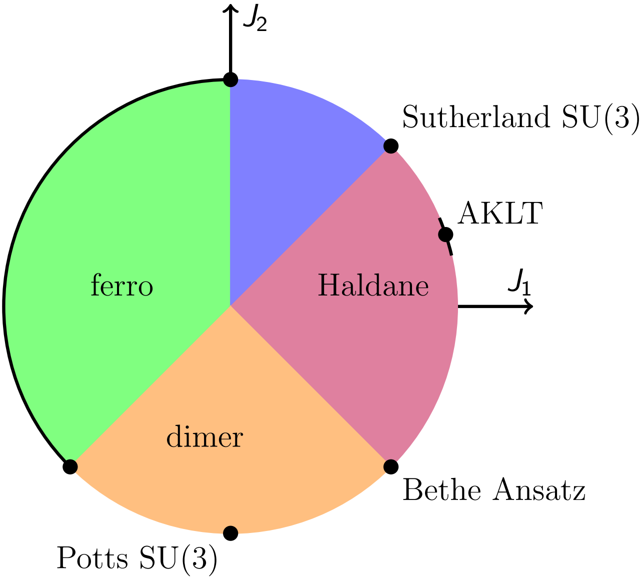
Antiferromagnetic spin-1 chain: $[1,L] \subset \mathbb{Z}$, $\mathcal{H}_x = \mathbb{C}^3$,

$$H_{[1,L]} = \sum_{x=1}^{L} \left(\frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2 \right) = \sum_{x=1}^{L} P^{(2)}_{x,x+1}$$

The ground state space of $H_{[1,L]}$ is 4-dimensional for all $L \geq 2$. In the limit of the infinite chain, the ground state is unique, has a finite correlation length, and there is a non-vanishing gap in the spectrum above the ground state (Haldane phase).

Theorem (Bachmann-N, CMP 2014)

There exists a curve of uniformly gapped Hamiltonians with nearest neighbor interaction $s \mapsto \Phi_s$ such that Φ_0 is the AKLT interaction and Φ_1 defines a PVBS model with $n_L = n_R = 1$ and a unique ground state of the infinite chain that is a product state.



$$H = \sum_x J_1 \mathbf{S}_x \cdot \mathbf{S}_{x+1} + J_2 (\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2$$

Symmetry protected phases

For a given system with G-symmetric interactions depending on a parameter s, we would like to find criteria to recognize that the model with $s = s_0$ is in a different gapped phase than with $s = s_1 \neq s_0$, meaning that the gap above the ground state necessarily closes for at least one intermediate value of s.

This is the same problem as before but restricted to a class of models with a given symmetry group (and representation) G.

Our goal is to find invariants, i.e., computable and, in principle, observable quantities that can be different at s_0 and s_1, only if the model is in a different ground state phase.

Consider

1. $\Gamma \subset \mathbb{Z}^\nu$, or another sufficient regular 'lattice', and an increasing and absorbing sequence of finite $\Lambda_n \uparrow \Gamma$. E.g., $\Gamma = \mathbb{Z}^\nu$, or Γ could be a half-space, or topological non-trivial with a boundary.
2. A family of models defined by a interaction Φ_s, $s \in [0, 1]$, and suppose
    - $s \mapsto \Phi_s(X)$ is differentiable and short-range
    - $\Phi_s(X)$ commutes with a local symmetry G, i.e. $[\Phi_s(X), \bigotimes_{x\in X} \pi(g)] = 0, g \in G$, π a unitary representation of G;
    - there is a uniform lower bound $\gamma > 0$ for the spectral gap above the ground state of $H_{\Lambda_n} := \sum_{X\subset\Lambda_n} \Phi_s(X)$, for all n.

Let $\tau_g(A) = \bigotimes_{x\in\Gamma} \pi(g)^* A\pi(g)$, for all $g \in G$, the action of the symmetry on observables Γ, and let σ_g^s denote the corresponding representation on the space spanned by the ground states: $\sigma_g^s(\omega) = \omega \circ \tau_g, \omega \in \mathcal{S}_s$.

Then, there exist quasi local automorphisms α_s such that

- $\alpha_s \circ \tau_g = \tau_g \circ \alpha_s$;
- $\mathcal{S}_s = \mathcal{S}_0 \circ \alpha_s$;
- $\sigma_g^s \cong \sigma_g^0$, for all $s \in [0, 1]$.

In other words:

Up to equivalence, the representation of G acting on the ground states of the model defined in Γ is constant within a gapped phase.

If Γ is, e.g., a half-space of a system with zero-energy edge modes, there will in general be a non-trivial representation on the space of edge states.

For two interesting classes of one-dimensional models this invariant, the representation of G given by σ_g, can be observed from the ground state in the bulk, i.e. in the model defined on $\mathbb{Z}$, i.e., Edge-Bulk correspondence. (Bachmann-N, JSP 2014).

This has also been done for some discrete symmetries for models with MPS ground states (Pollmann & Turner, PRB 2012) and for certain SU(N) spin chains (Duivenvoorden & Quella, PRB 2012) and in a different way for MPS states by Haegeman, Perez-Garcia, Cirac, & Schuch (PRL 2012).

Symmetry protected phases in 1 dimension: Half-chains

Consider $\Gamma = [1, +\infty) \subset \mathbb{Z}$, and translation-invariant models defined by a nearest-neighbor interaction $h(s)$, $s \in [0, 1]$. Suppose

- $s \mapsto h(s)$ is differentiable;
- $h(s)$ commutes with a local symmetry G, i.e. $[h(s), \pi(g) \otimes \pi(g)] = 0, g \in G$, π a representation of G;
- there is a uniform lower bound $\gamma > 0$ for the spectral gap above the ground state of $\sum_{x=1}^{L-1} h_{x,x+1}(s)$, for all $L \geq 2$.

Let $\tau_g(A) = \bigotimes_{x \in \Gamma} \pi(g)^* A \pi(g)$, for all $g \in G$, the action of the symmetry on observables of the half-chain, and let σ_g^s denote the corresponding representation on the space spanned by the ground states: $\sigma_g^s(\omega) = \omega \circ \tau_g, \omega \in \mathcal{S}_s$.

Models to keep in mind: antiferromagnetic chains in the Haldane phase and generalizations. Unique ground state with a spectral gap and an unbroken continuous symmetry.

Let S^i_x, $i = 1, 2, 3$, $x \in \mathbb{Z}$, denote the ith component of the spin at site x. Claim: one can define

$$\sum_{x=1}^{+\infty} S^i_x,$$

as s.a. operators on the GNS space of the ground state on $\mathbb{Z}$ and they generate a representation of $SU(2)$ that is characteristic of the gapped ground state phase.
We can prove the existence of these excess spin operators for two classes of models:
1) models with a random loop representation;
2) models with a matrix product ground state (MPS).

E.g.: Frustration-free chains with an $SU(2)$ invariant MPS ground state

$$H = \sum_x h_{x,x+1}$$

Ground state is defined in terms of an isometry V, which intertwines two representations of $SU(2)$:

$$Vu_g = (U_g \otimes u_g)V, \quad g \in SU(2).$$

E.g., in the AKLT chain U_g is the spin-1 representation and u_g is the spin-$1/2$ representation of $SU(2)$, corresponding to the well-known spin $1/2$ degrees of freedom at the edges.
Let $k = \dim(u_g)$. The transfer operator is defined by

$$\mathbb{E}(B) = V^*(\mathbb{1} \otimes B)V, B \in M_k.$$

If ω is a G-invariant, pure, translation-invariant finitely correlated state generated by the intertwiner V, one can assume that 1 is the unique eigenvalue of maximal modulus of $\mathbb{E}$, and that it is simple (Fannes-N-Werner, JFA 1994).

Let $S = (S^1, S^2, S^3)$ be the vector of generators of U_g, and write $U_g = \exp(ig \cdot S)$. Define

$$S^+(L) = \sum_{x=1}^{L^2} f_L(x-1) S_x,$$

where $f_L : \mathbb{Z}^+ \to \mathbb{R}$ is given by

$f_L(mL+n) = 1-m/L$, if $m, n \in [0, L-1]$, and $f_L(x) = 0$, if $x \geq L^2$.

Then, $U_g^+(L) = \exp(igS^+(L))$ is an observable and use the same notation for its representative on the GNS Hilbert space, $\mathcal{H}_\omega$, of ω.

Theorem

Let ω be as above. Then, the strong limit

$$U_g^+ = \lim_{L\to\infty} e^{ig\cdot S^+(L)}$$

exists and defines a strongly continuous unitary representation of G on $\mathcal{H}_\omega$.

The representation of U_g^+ is an invariant

Summary of the proof: $U_g^+|_{\pi_\omega(\mathcal{A}_{(-\infty,0]})\Omega_\omega} \cong (\oplus u_g)^\infty$.
(i) First consider the model on the half-infinite chain. The space of ground states transforms as u_g under the action of $SU(2)$. We call this the edge spin representation. We proved that, in general, along a curve of models with a non-vanishing gap, the edge representation is constant.
(ii) On the infinite chain, we showed that the excess spin representation is well-defined.
(iii) One can verify that on the subspace of the GNS Hilbert space of the infinite-chain ground state consisting of the ground state of the Hamiltonian of the half-infinite chain, acts as (an infinite number of copies of) u_g.

This is also shows that u_g is experimentally observable.

Concluding Remarks

- Version of edge-bulk correspondence is valid in the symmetry protected case for certain classes of models such as FF chains with parent hamiltonians.
- There are infinitely many inequivalent $SU(2)$ and translation invariant gapped ground state phases of integer spin chains.
- You can classify the types of topological order by your favorite method (homotopy of occupied bands, cohomology of symmetry group), but you should not expect the ordered phases in realistic models to necessarily be continuously connected to the toy model representatives; other quantum phase transitions affect basic properties of the ground state, the topological ordered "phase", could be a collection of disconnected pieces.
- 2 and more dimensions!!!

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