Universal operations on Hochschild cplx on alg over props

(partly j.w. Westerland)

\[ A : \text{dg-alg} \quad (\text{could be } A_{\Sigma} \text{-alg}) \]

\[ C^*(A, A) = \bigoplus_{n \geq 1} A^\otimes_n \quad (\text{or reduced ver } \bigoplus_{n=0} A^\otimes_{\Sigma}) \]

with \[ d = d_A + d_A \]

Operations \[ C^*(A, A) \otimes n \to C^*(A, A) \otimes m \]

linear maps, also consider degrees.

the space of operations becomes a chain cplx

by \[ (d_{\text{op}})(x) = y(dx) + dy(x) \]

sign (cancellation \( 6 \text{oper.} x \otimes C^*(A, A) \otimes n \to C^*(A, A) \otimes m \)

is a chain map.

Questions 1) What's the cplx of "all" natural operations?

2) how does "extra structure" (e.g., Frobenius, comm)

give structures on \[ C^*(A, A) \]

3) where is the cap product?

4) what about graph cplxes?

PROP = "category of operations"

category with \( \text{obj set } N \). \( \text{Mor}(n, m) = \text{operations} \)

with \( n \) input, \( m \) output.

composition \( \text{compos. of operations} \)

sum (mon.str.): juxtaposition of ops. \( n' \leq m \)
Example: category of punctured surfaces.

\[ \text{Mor}(n, m) = \mathbb{E} \text{ top. types of surf. w/ n+m bdry comp.s} \]

* associative alg \( \Rightarrow M \in \text{Mor}(2, 1) \).

\[ \text{Mor}(n, m) = \mathbb{E}\text{n(\_)}\_m \]

**PROP with multiplications** = **PROP w/ specified**

\[ m \in \text{Mor}(2, 1) \]

**Theorem 1**: \( P: \text{PROP with multiplication } M \in P(2, 1) \)

If \( A \) is a \( P \)-alg. (i.e. \( \exists P(n, m) \otimes A \otimes n \rightarrow A \otimes m \)

chain map. compat w/ str. in \( P \)), then

\( \text{C}*(A, A) \) w.r.t. \( M \) is a "Hoch(P)" alg.,

i.e. \( \exists \text{Hoch}(P)(n, m) \otimes \text{C}*(A, A) \otimes n \rightarrow \text{C}*(A, A) \otimes m \)

where \( \text{Hoch}(P)(n, m) = \bigoplus_{j_1, \ldots, j_n, k_1, \ldots, k_m} P(j_1 + \cdots + j_n, k_1 + \cdots + k_m) \)

with \( d = d_P + b + \cdots + b^{[*]i} \) from \( M \) acting on "input"

from \( M \) acting on "output".

Hoch(P) is the opx of all operations made out of

* operations from \( P \) (e.g. mult., inverting units, etc.)

* permuting the \( A \)'s.
More precisely, 
\[ C^*(A, A) \otimes \mathbb{N} \quad \overset{\varphi_A}{\longrightarrow} \quad C^*(A, A) \otimes \mathbb{N} \]

is given by 
\[ \varphi_A = \{ (\varphi_A)_{j_1, \ldots, j_n} : A^{\otimes j_1 + \cdots + j_n} \otimes k_{j_1 + \cdots + j_n} \} \]

Thm 2. \( \mathcal{P}_\text{Alg} := \text{Fun} \otimes (\mathcal{P}, \text{dgVect}) \xrightarrow{C^*} \text{dgVect} \rightarrow \text{Fun} \otimes (\mathcal{P}, \text{dgVect}) \)

\[ \text{such that } \text{Hoch}(\mathcal{P})(n, m) = \text{Hom}\left( C^n, C^m \right) \text{Fun}(\text{Func}^{-1}, \text{dg}) \]

Do we understand 
\[ \text{Hom}_{\text{Fun}(\text{Func}^{-1}, \text{dg})} \left( C^n, C^m \right) \xrightarrow{C^*} \text{Hom}_{\text{Func}(\text{Func}} \otimes \text{dg}) \left( C^n, C^m \right) \]

Thm 3. With the "completion" 
\[ \mathcal{P}(n, m) = \text{Hom}(\mathbb{N} \otimes \mathbb{N}) \]

\[ \text{Hom}(C(-,-) \otimes \mathbb{N}, C(-,-) \otimes \mathbb{N}) = \text{Hoch}(\mathcal{P})(n, m) \]

Example / application

1) Unital associative algs.: we have \( B : C^*(A, A) \to C^*(A, A) \)

Fact. \( \text{Hoch}(\text{Ass}) \) is generated by \( B \) and \( \text{id} \).

Cap product 
\[ C^*(A, A) \otimes C^*(A, A) \xrightarrow{\cap} C^*(A, A) \]

this def's a map 
\[ C^*(A, A) \to \text{Hoch}(\text{End}(A))(1, 1) \]

\( \text{End}(A)(n, m) = \text{Hom}(A^{\otimes n}, A^{\otimes m}) \).
(suite) 2) symmetric Frobenius algebras.

Correct PROP: surfaces with intervals in $\partial Y$. 

\[ (\text{isoclasses of trivalent}) \text{ ribbon graphs} \ n + m \text{ legs} = P. \]

\[ \text{PreHoch} (P) = \bigoplus P_{G, k_1 + \ldots + k_m} \otimes k_1 \otimes \ldots \otimes k_m \]

dege shift by $k_p - 1$

\[ l_k \leftrightarrow \tilde{\gamma}_k \]

Present PreHoch ($P$) by graph elements of Hochschild diff

Kontsevich-Soibelman: $\exists$ "cylinder cplx" $C_x / *$

$C_x^k \oplus C_* (A, A)^{\otimes k} \to \text{Hoch} (\text{End}(A))(1, 1)$

$\xi, \xi, \psi, \psi$