Higher Structures on Modules over Operads

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Let $A$ be an associative algebra over a commutative ring $k$, set $A^e := A \otimes_k A^{\text{op}}$, and let $M$ be an $A^e$-module.

- Define the Hochschild chain and cochain complexes (Hochschild, 1945) with coefficients in $M$ as

$$C^\bullet(A, M) = \text{Hom}_k(A^{\otimes_k \cdot}, M), \quad C_\bullet(A, M) := M \otimes_k A^{\otimes_k \cdot},$$

with respective differentials $\beta$ and $b$ that are defined using the multiplication in $A$ and the $A^e$-module structure on $M$, giving rise to Hochschild cohomology $H^\bullet(A, M)$ and homology $H_\bullet(A, M)$.

- $H^\bullet(A, M)$ is an interesting object (depending on your interests) since it classifies infinitesimal deformations of the multiplicative structure of $A$. 

Higher structures on Hochschild (co)homology

- Gerstenhaber (1963) discovered that $H^\bullet(A, M)$ carries more structure:
  - that of a graded commutative multiplication $\cdot$;
  - that of a graded Lie bracket $[,]$ of degree $-1$;
  - such that $[,]$ is a graded derivation of $\cdot$.

For this to hold, $M$ needs to satisfy certain conditions.

- Rinehart (1963), Daletski-Gel’fand-Tsygan (1989), Nest-Tsygan (1998), and partially others (Connes, Getzler, Goodwillie) noticed (for $M := A$) that there is even more structure to be unveiled on the pair $(H^\bullet(A, A), H_*^\bullet(A, A))$:
  - a graded $H^\bullet(A, A)$-module structure $\iota$ on $H_*^\bullet(A, A)$;
  - a graded $H^\bullet(A, A)$-Lie algebra module structure $L$ on $H_*^\bullet(A, A)$;
  - such that with the cyclic differential $B$ one has the homotopy formula
    $$L = [B, \iota].$$

Some of the above structure can be already seen on the (co)chain level $(C^\bullet(A, A), C_*^\bullet(A, A))$, others need a “homotopic correction”.

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Digression: cyclic objects

We briefly recall the concept of cyclic objects and cyclic homology:

- Let $\mathcal{C}$ be a category and $\Lambda$ Connes’ cyclic category (a combination of the simplicial category and the cyclic groups). A **para-cyclic object** in $M$ is a functor $X: \Lambda^{\text{op}} \to \mathcal{C}$, that is, a simplicial object $(X_\bullet, d_\bullet, s_\bullet)$ together with morphisms $t_n : X_n \to X_n$ subject to

$$d_i t_n = \begin{cases} t_{n-1} d_{i-1} & \text{if } 1 \leq i \leq n \\ d_n & \text{if } i = 0 \end{cases} \quad \quad s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{if } 1 \leq i \leq n \\ t_{n+1}^2 s_n & \text{if } i = 0. \end{cases}$$

- A para-cyclic object is called **cyclic** if $t_n^{n+1} = \text{id}$.

- Define Hochschild operator, norm operator, extra degeneracy:

$$b := \sum_{j=0}^{n} (-1)^j d_j, \quad N := \sum_{j=0}^{n} (-1)^n t_{n+1}^j, \quad s_{-1} := t_{n+1} s_n,$$

and define **Connes’ cyclic operator** or **Connes’ coboundary**:

$$B := (1 - (-1)^n t_{n+1}) s_{-1} N.$$
These operators fulfill $B^2 = 0$, $Bb + bB = 0$, and $b^2 = 0$, hence each cyclic object gives rise to a mixed complex: define $BC_\bullet \bullet X$ by $BC_{p,q}(X) := X_{q-p}$ for $0 \leq p \leq q$ and zero otherwise:

The homology of the zeroth column is the Hochschild homology

$$HH_\bullet(X) := H_\bullet (C_\bullet X, b),$$

whereas cyclic homology is defined to be the homology of the total complex:

$$HC_\bullet(X) := H_\bullet (\text{Tot } BC_\bullet \bullet X, b + B).$$
Gerstenhaber algebras, more precisely

**Definition**

Let $k$ be a commutative ring. A **Gerstenhaber algebra** over $k$ is a graded commutative $k$-algebra $(V, \smile)$

$$V = \bigoplus_{p \in \mathbb{N}} V^p, \quad \alpha \smile \beta = (-1)^{pq} \beta \smile \alpha \in V^{p+q}, \quad \alpha \in V^p, \beta \in V^q$$

with a graded Lie bracket $\{\cdot, \cdot\} : V^{p+1} \otimes_k V^{q+1} \to V^{p+q+1}$ on the desuspension

$$V[1] := \bigoplus_{p \in \mathbb{Z}} V^{p+1}$$

for which all operators $\{\gamma, \cdot\}$ satisfy the graded Leibniz rule

$$\{\gamma, \alpha \smile \beta\} = \{\gamma, \alpha\} \smile \beta + (-1)^{pq} \alpha \smile \{\gamma, \beta\}, \quad \gamma \in V^{p+1}, \alpha \in V^q.$$
Gerstenhaber modules

Definition

A **Gerstenhaber module** over $V$ is a graded $(V, \triangleright)$-module $(\Omega, \triangleright)$,

$$\Omega = \bigoplus_{n \in \mathbb{Z}} \Omega_n, \quad \iota_{\alpha} x := \alpha \triangleright x \in \Omega_{n-p}, \quad \alpha \in V^p, x \in \Omega_n$$

with a representation of the graded Lie algebra $(V[1], \{\cdot, \cdot\})$

$$\mathcal{L} : V^{p+1} \otimes_k \Omega_n \to \Omega_{n-p}, \quad \alpha \otimes_k x \mapsto \mathcal{L}_\alpha(x)$$

which satisfies for $\alpha \in V^{p+1}, \beta \in V^q, x \in \Omega$ the mixed Leibniz rule

$$\beta \triangleright \mathcal{L}_\alpha(x) = \{\beta, \alpha\} \triangleright x + (-1)^{pq} \mathcal{L}_\alpha(\beta \triangleright x)$$
Batalin-Vilkovisky modules $=$ differential calculi

**Definition**

A Gerstenhaber module is **Batalin-Vilkovisky** if there is a $k$-linear differential

$$B : \Omega_n \to \Omega_{n+1}, \quad BB = 0$$

such that $L_\alpha$ is for $\alpha \in V^p$ given by the **homotopy formula**

$$L_\alpha(x) = B(\alpha \lrcorner x) - (-1)^p \alpha \lrcorner B(x).$$

A pair $(V, \Omega)$ of a Gerstenhaber algebra and of a Batalin-Vilkovisky module over it is also called a **(noncommutative) differential calculus**.
The very classical example

- $X$ a compact smooth manifold, $k := \mathbb{C}$. Then

$$V := \bigwedge_A \text{Der}_\mathbb{C}(A), \quad A := \mathcal{C}^\infty(X, \mathbb{C})$$

is a Gerstenhaber algebra w.r.t. the **Schouten-Nijenhuis bracket** which for vector fields is just the commutator.

- The differential forms $\Omega(X)$ on $X$ are a Gerstenhaber module over $V$. The module structure is given by the **contraction** $\iota$ of a differential form by a multivector field and the action $\mathcal{L}$ is the **Lie derivative** of a differential form along a multivector field.

- The **Cartan homotopy** or **magic formula**, that is,

$$\mathcal{L} = [\iota, d]$$

says that this Gerstenhaber module is BV with respect to the **exterior derivative** $\mathcal{B} := d$. 
The now classical example

- Let $A$ be an associative algebra over a field $k$ (or at least let $A$ be projective over a commutative ring $k$). Then its **Hochschild cohomology**

\[ V := H^\bullet(A, A) = \text{Ext}_{A^e}^\bullet(A, A), \quad A^e := A \otimes_k A^{\text{op}} \]

is canonically a Gerstenhaber algebra (Gerstenhaber’s original example 1963).

- The **Hochschild homology**

\[ \Omega := H_\bullet(A, A) = \text{Tor}_{A^e}^\bullet(A, A) \]

is canonically a BV module over $V$ (Rinehart 1963, Connes, Getzler, Goodwillie, and ultimately Nest-Tsygan in the 80s and 90s).
The (sort-of) universal example

**Theorem (K.-Krähmer 2012)**

*If $U$ is a left Hopf algebroid which is right projective over its base algebra $A$ and $M$ is a stable anti Yetter-Drinfel’d (SaYD) module over $U$, then*

$$\left( \text{Ext}_U^\bullet(A, A), \text{Tor}_U^\bullet(M, A) \right)$$

*carries a canonical structure of a noncommutative differential calculus.*

This contains the examples above, but also applies to **groups, groupoids, Lie algebras, Lie algebroids, Poisson algebras**, etc. An extension is

**Theorem (K., 2013)**

*If $(U, A)$ and $M$ is as above, and $N$ a braided commutative Yetter-Drinfel’d algebra such that $M \otimes_A N$ is SaYD, then*

$$\left( \text{Ext}_U^\bullet(A, N), \text{Tor}_U^\bullet(M, N) \right)$$

*carries a canonical structure of a noncommutative differential calculus.*
Question

How to encode this algebraic structure as compact as possible, possibly even without the bialgebroid framework?

- For example, one already knows from Gerstenhaber-Schack, McClure-Smith (and probably others) that if $\mathcal{O}$ is an **operad with multiplication**, then it gives rise to a cosimplicial structure and the respective cohomology $H^\bullet(\mathcal{O})$ carries the structure of a Gerstenhaber algebra.

- If one furthermore (in some sense) dualises the notion of a cyclic operad (in a sense to be specified below), one obtains:

**Theorem (K., 2013)**

Let $\mathcal{O}$ be an operad (in $k$-$\text{Mod}$) with multiplication and $\mathcal{M}$ a cyclic unital (comp) $\mathcal{O}$-module. Then $\mathcal{M}$ carries the structure of a cyclic $k$-module and the pair $(H^\bullet(\mathcal{O}), H_\bullet(\mathcal{M}))$ defines a noncommutative differential calculus.
Operads: the fundamental example

Consider the space \( \mathcal{O} := \text{Hom}_k(A^\otimes k\cdot, A) \) of Hochschild cochains with values in \( A \).

- For \( \phi \in \mathcal{O}^p \) and \( \psi \in \mathcal{O}^q \), define the operation

  \[
  \phi \circ_i \psi(a_1, \ldots, a_{p+q-1}) := \phi(a_1, \ldots, a_{i-1}, \psi(a_i, \ldots, a_{i+q-1}), a_{i+q}, \ldots, a_{p+q-1}),
  \]

  that is, insertion of \( \psi \) into \( \phi \) at the \( i \)th slot.

- This apparently works replacing \( A \) by any vector space \( V \), but in the Hochschild case there is a **distinguished element** \( \mu \in \mathcal{O}^2 \) given by

  \[
  \mu(a, b) := ab, \quad a, b \in A,
  \]

  the multiplication in \( A \), which will serve in a moment.

- The operation is not (strictly speaking) associative, but sort of in a more general sense:
Associativity behaviour

Figure: Parallel composition axiom
Figure: Sequential composition axiom
Operads: formal definition

**Definition ("Partial" definition)**

A (non-$\Sigma$, unital) operad (in $k$-$\text{Mod}$) is a sequence $\{O^n\}_{n \geq 0}$ of $k$-modules with an identity element $1 \in O^1$ and $k$-bilinear operations $o_i : O^p \otimes O^q \rightarrow O^{p+q-1}$ such that for $\varphi \in O^p$, $\psi \in O^q$, and $\chi \in O^r$:

$$\varphi \circ_i \psi = 0 \quad \text{if } p < i \quad \text{or} \quad p = 0,$$

$$\varphi \circ_i \psi \circ_j \chi = \begin{cases} 
(\varphi \circ_j \chi) \circ_{i+r-1} \psi & \text{if } j < i, \\
\varphi \circ_i (\psi \circ_{j-i+1} \chi) & \text{if } i \leq j < q+i, \\
(\varphi \circ_{j-q+1} \chi) \circ_i \psi & \text{if } j \geq q+i,
\end{cases}$$

$$\varphi \circ_i 1 = 1 \circ_i \varphi = \varphi \quad \text{for } i \leq p$$

The operad $O$ is called **operad with multiplication** if there exists a multiplication $\mu \in O^2$ and a unit $e \in O^0$ such that

$$\mu \circ_1 \mu = \mu \circ_2 \mu, \quad \text{and} \quad \mu \circ_1 e = \mu \circ_2 e = 1.$$
Gerstenh. algebras from operads w. multiplication

**Theorem (Gerstenhaber-Schack/McClure-Smith)**

*Each operad with multiplication defines a cosimplicial $k$-module. Its cohomology forms a Gerstenhaber algebra.*

The explicit structure maps read as follows: for $\varphi \in O^p$, $\psi \in O^q$, set

$$\varphi \bar{\circ} \psi := \sum_{i=1}^{p} (-1)^{|q||i|} \varphi \circ_i \psi \in O^{p+q-1}, \quad |n| := n - 1,$$

and define their **Gerstenhaber bracket** by

$$\{\varphi, \psi\} := \varphi \bar{\circ} \psi - (-1)^{|p||q|} \psi \bar{\circ} \varphi.$$

The graded commutative product is given by the **cup product**

$$\varphi \smile \psi := (\mu \circ_2 \varphi) \circ_1 \psi \in O^{p+q}.$$

Finally, the coboundary of the cosimplicial $k$-module results as

$$\delta \varphi = \{\mu, \varphi\},$$

and then the triple $(O, \delta, \smile)$ forms a **DG algebra**.
As is well-known, the cup product is only commutative up to homotopy. More precisely,

\[
(-1)^{p-1} \varphi \bar{\circ} \delta \psi - (-1)^{p-1} \delta (\varphi \bar{\circ} \psi) + \delta \varphi \bar{\circ} \psi = \varphi \triangledown \psi - (-1)^{pq} \psi \triangledown \varphi.
\]

Thinking of \( \triangledown =: \triangledown_0 \) and \( \bar{\circ} =: \bar{\circ}_1 \), this is the Steenrod relation in degree 1.
(Comp) modules over operads: the fundamental example

For an associative $k$-algebra $A$, define

$$\mathcal{M} := C_\bullet(A) := A^{\otimes_k \bullet + 1}$$

and

$$\mathcal{O} := C^\bullet(A) := \text{Hom}_k(A^{\otimes_k \bullet}, A).$$

There is an obvious collection of maps $\bullet_i : \mathcal{O} \otimes_k \mathcal{M} \to \mathcal{M}$ given by

$$\varphi \bullet_i (a_0, \ldots, a_n) := (a_0, \ldots, a_{i-1}, \varphi(a_i, \ldots, a_{i+p-1}), a_{i+p}, \ldots, a_n),$$

for $i = 1, \ldots, n - p + 1$ and $n \in \mathbb{N}$, which satisfies similar “associativity” relations as those for operads:
**Definition**

A **comp** or **opposite module** $\mathcal{M}$ over an operad $\mathcal{O}$ (called $\mathcal{O}$-**module** for short) is a sequence $\{\mathcal{M}_n\}_{n \geq 0} \in k\text{-Mod}$ together with $k$-linear operations

$\bullet_i : \mathcal{O}^p \otimes_k \mathcal{M}_n \to \mathcal{M}_{n-p+1}, \quad \text{for } i = 1, \ldots, n-p+1, \quad 0 \leq p \leq n,$

and zero if $p > n$, and satisfying for $\varphi \in \mathcal{O}^p$, $\psi \in \mathcal{O}^q$, and $x \in \mathcal{M}_n$

$$\varphi \bullet_i (\psi \bullet_j x) = \begin{cases} 
\psi \bullet_j (\varphi \bullet_{i+q-1} x) & \text{if } j < i, \\
(\varphi \circ_{j-i+1} \psi) \bullet_i x & \text{if } j - p < i \leq j, \\
\psi \bullet_{j-p+1} (\varphi \bullet_i x) & \text{if } 1 \leq i \leq j - p,
\end{cases}$$

where $p > 0$, $q \geq 0$, $n \geq 0$. An $\mathcal{O}$-module is called **unital** if

$$1 \bullet_i x = x, \quad \text{for } i = 1, \ldots, n, \text{ for all } x \in \mathcal{M}_n,$$
Comp modules depicted

Keeping the fundamental example in mind, a comp module could be depicted as:

Figure: Comp modules
Cyclic comp modules: the fundamental example

For

\[ \mathcal{M} := C_\bullet(A) \] and \[ \mathcal{O} := C^\bullet(A) \]

as above, apparently even the map

\[ \varphi \bullet_0 (a_0, \ldots, a_n) := (\varphi(a_0, \ldots, a_{p-1}), a_p, \ldots, a_n) \]

makes sense. With the usual cyclic operator

\[ t(a_0, \ldots, a_n) := (a_n, a_0, \ldots, a_{n-1}) \]

one observes the formula

\[ t(\varphi \bullet_i (a_0, \ldots, a_n)) = \varphi \bullet_{i+1} t(a_0, \ldots, a_n), \]
Cyclic $\mathcal{O}$-modules

Crucial is then the following notion dual to that of a cyclic operad:

**Definition (K., 2013)**

A **para-cyclic unital (comp) $\mathcal{O}$-module** is a unital (comp) $\mathcal{O}$-module $\mathcal{M}$ as before, equipped with two additional structures:

- an **extra** ($k$-linear) comp module map

\[ \bullet_0 : \mathcal{O}^p \otimes_k \mathcal{M}_n \to \mathcal{M}_{n-p+1}, \quad 0 \leq p \leq n+1, \]

declared to be zero if $p > n + 1$, and such that the above associativity relations extends to $i = 0$;

- a morphism $t : \mathcal{M}_n \to \mathcal{M}_n$ such that for $\varphi \in \mathcal{O}^p$ and $x \in \mathcal{M}_n$

\[ t(\varphi \bullet_i x) = \varphi \bullet_{i+1} t(x), \quad i = 0, \ldots, n-p. \]

A para-cyclic $\mathcal{O}$-module is called **cyclic** if $t^{n+1} = \text{id}$ is true on $\mathcal{M}_n$. 
Cyclic comp modules depicted

Figure: The relation $t(\varphi \bullet_i x) = \varphi \bullet_{i+1} t(x)$ for cyclic comp modules
Homology of cyclic $\mathcal{O}$-modules

Proposition (K., 2013)

Let $(\mathcal{O}, \mu)$ be an operad with multiplication. A cyclic unital $\mathcal{O}$-module defines a cyclic $k$-module $(d_\bullet, s_\bullet, t_\bullet)$, where for $x \in M_n$

\[
\begin{align*}
  d_i(x) &= \mu \bullet_i x, & i = 0, \ldots, n - 1, \\
  d_n(x) &= \mu \bullet_0 t(x), \\
  s_j(x) &= e \bullet_{j+1} x, & j = 0, \ldots, n.
\end{align*}
\]

As usual, one defines the Hochschild or simplicial boundary operator $b := \sum_{i=0}^{n} (-1)^i d_i$, explicitly here

\[
b := \sum_{i=0}^{n-1} (-1)^i \mu \bullet_i x + (-1)^n \mu \bullet_0 t(x),
\]

and the norm operator, the extra degeneracy, and the cyclic differential

\[
N := \sum_{i=0}^{n} (-1)^i n t^i, \quad s_{-1} := t s_n, \quad B = (\text{id} - t) s_{-1} N.
\]
Homology of cyclic \( \mathcal{O} \)-modules

**Definition**

For any cyclic unital \( \mathcal{O} \)-module \( \mathcal{M} \) over an operad \((\mathcal{O}, \mu)\) with multiplication, we call the homology of \((\mathcal{M}_\bullet, b)\), denoted by \( H_\bullet(\mathcal{M}) \), its **(Hochschild) homology**, and the homology of the mixed complex \((\mathcal{M}_\bullet, b, B)\), denoted \( HC_\bullet(\mathcal{M}) \), the **cyclic homology** of \( \mathcal{M} \).

- We did not call in a purely random manner the action \( \bullet_0 \) an “extra” comp module map: one computes

  \[
  e \bullet_0 x = s_{-1}(x)
  \]

  for \( x \in \mathcal{M}_n \), hence the extra degeneracy. With this, one obtains for \( x \in \tilde{\mathcal{M}}_n \)

  \[
  B(x) = \sum_{i=0}^{n} (-1)^{in} e \bullet_0 t^i(x).
  \]
Cyclic $\mathcal{O}$-modules as Gerstenhaber modules

Definition (K., 2013)

Let $(\mathcal{O}, \mu)$ be an operad with multiplication and $\mathcal{M}$ a cyclic unital $\mathcal{O}$-module. Define

- the **cap product** of $\varphi \in \mathcal{O}^p$ with $x \in \mathcal{M}_n$ by

  \[ \iota_{\varphi} := \varphi \curvearrowright \cdot : \mathcal{M}_n \to \mathcal{M}_{n-p}, \quad \varphi \curvearrowright x := (\mu \circ_2 \varphi) \bullet_0 x. \]

- the **Lie derivative** $\mathcal{L}_{\varphi} : \mathcal{M}_n \to \mathcal{M}_{n-p+1}$ of $x \in \mathcal{M}_n$ along $\varphi \in \mathcal{O}^p$ with $p < n + 1$ by

  \[ \mathcal{L}_{\varphi} x := \sum_{i=1}^{n-p+1} (-1)^{\xi_i^p} \varphi \bullet_i x + \sum_{i=1}^{p} (-1)^{\xi_{p,i}^n} \varphi_0 \bullet_i t^{i-1}(x), \]

where $\xi_i^p$ and $\xi_{p,i}^n$ are sign functions.

Compare the formal similarity of the cap with the cup product.
DG module and DG Lie algebra module structures

Dual to the fact that \((\mathcal{O}, \delta, \hookrightarrow)\) is a DG algebra, one has:

**Theorem (K., 2013)**

The triple \((\mathcal{M}_{-\bullet}, b, \hookrightarrow)\) defines a **left DG module** over \((\mathcal{O}, \delta, \hookrightarrow)\),

\[
\iota_\varphi \iota_\psi = \iota_{\varphi \hookrightarrow \psi}, \quad [b, \iota_\varphi] = \iota_{\delta \varphi},
\]

where \(\varphi \in \mathcal{O}^p, \psi \in \mathcal{O}\). The Lie derivative defines a **DG Lie algebra representation** of \((\mathcal{O}[1], \{.,.\})\) on \(\mathcal{M}_{-\bullet}\),

\[
[L_\varphi, L_\psi] = L_{\{\varphi,\psi\}}.
\]

Moreover, the simplicial differential on \(\mathcal{C}_\bullet(\mathcal{M})\) is given by the Lie derivative along the multiplication:

\[
b = -L_\mu \quad \text{and} \quad [b, L_\varphi] + L_{\delta \varphi} = 0.
\]

All proven by tedious computations.
The Gerstenhaber module

By the preceding formulae, both \( \iota_\varphi \) and \( L_\varphi \) descend to well defined operators on the simplicial homology \( H_\bullet(\mathcal{M}) \) as soon as \( \varphi \) is a cocycle. In this case, together with the preceding theorem, the operations \( \iota \) and \( L \) turn \( H_\bullet(\mathcal{M}) \) into a module over the Gerstenhaber algebra \( H^\bullet(\mathcal{O}) \):

**Theorem (K., 2013)**

For any two cocycles \( \varphi \in \mathcal{O}^p, \psi \in \mathcal{O}^q \), the induced maps

\[
L_\varphi : H_\bullet(\mathcal{M}) \to H_{\bullet-p+1}(\mathcal{M}), \quad \iota_\psi : H_\bullet(\mathcal{M}) \to H_{\bullet-q}(\mathcal{M})
\]

satisfy

\[
[\iota_\psi, L_\varphi] = \iota_{\{\psi, \varphi\}}.
\]

A bit trickier to prove involving a recursive argument plus still a bunch of truly unpleasant computations.
The Batalin-Vilkovisky module structure

We now give a sort of homotopy formula relating the cyclic differential with the Lie derivative and the cap product on the chain level. Since here the full cyclic bicomplex is “seen”, $ι_ϕ$ needs a “cyclic correction” $S_ϕ$, and we call the sum $l_ϕ := ι_ϕ + S_ϕ$ the **cyclic cap product**. Define for $0 ≤ p ≤ n$

$$S_ϕ : M_n → M_{n−p+2},$$

$$S_ϕ := \sum_{j=1}^{n−|p|} \sum_{i=j}^{n−|p|} (-1)^{\vartheta_{j,i}^{n,p}} e \bullet_0 (ϕ \bullet_i t^{j−1}(x)),$$

with a certain sign function $\vartheta_{j,i}^{n,p}$. If $p > n$, put $S_ϕ := 0$. 

The Batalin-Vilkovisky module structure

**Theorem (K., 2013)**

For any cochain $\varphi \in \tilde{O}^\bullet$ in the normalised cochain complex $\tilde{O}^\bullet$, the Cartan-Rinehart homotopy formula

$$L_\varphi = [B + b, \iota_\varphi + S_\varphi] - \iota_\delta \varphi - S_\delta \varphi$$

$$= [B + b, I_\varphi] - I_\delta \varphi$$

holds on the normalised chain complex $\tilde{M}_\bullet$. For cocycles, this simplifies on homology to

$$L_\varphi = [B, \iota_\varphi].$$
Examples I: associative algebras

For an associative $k$-algebra $A$, define

$$M := C_\bullet(A) := A^{\otimes_k \bullet+1}$$

and

$$O := C^\bullet(A) := \text{Hom}_k(A^{\otimes_k \bullet}, A).$$

$$(\varphi \circ_i \psi)(a_1, \ldots, a_{p+q-1}) := \varphi(a_1, \ldots, a_{i-1}, \psi(a_i, \ldots, a_{i+q-1}), a_{i+q}, \ldots, a_{p+q-1}),$$

$$\mu(a \otimes_k b) := ab, \quad 1 := \text{id}_A(\cdot), \quad e := 1_A,$$

$$\varphi \bullet_i (a_0, \ldots, a_n) := (a_0, \ldots, a_{i-1}, \varphi(a_i, \ldots, a_{i+p-1}), a_{i+p}, \ldots, a_n),$$

for $i = 1, \ldots, n - p + 1$. This gives a $C^\bullet(A)$-module structure on $C_\bullet(A)$. Extending this formula in the obvious way to $i = 0$ and with the usual cyclic operator

$$t(a_0, \ldots, a_n) := (a_n, a_0, \ldots, a_{n-1}).$$

this gives a cyclic $C^\bullet(A)$-module. The operators $\circ, \bowtie, B, L_\varphi, S_\varphi$ are those found in the work of Daletski, Gel’fand, Nest, Tamarkin, Tsygan and those on which the construction of Kontsevich-Soibelman is based.
Examples I.b: twisted Calabi-Yau algebras

Extending the previous example, consider for a $\sigma \in \text{Aut}(A)$ the algebra $A$ itself as an $(A, A)$-bimodule $A_\sigma$ via $a \triangleright x \triangleleft b := ax\sigma(b)$ for $a, b, x \in A$, and set

$$\mathcal{M} := C_\bullet(A, A_\sigma) := A_\sigma \otimes_k A \otimes_k \bullet$$

and

$$\mathcal{O} := C^\bullet(A) := \text{Hom}_k(A \otimes_k \bullet, A).$$

In case $\sigma$ is semisimple (diagonalisable), that is, if there is a subset $\Sigma \subseteq k \setminus \{0\}$ and a decomposition of $k$-vector spaces

$$A = \bigoplus_{\lambda \in \Sigma} A_\lambda, \quad A_\lambda = \{a \in A \mid \sigma(a) = \lambda a\},$$

then $(H^\bullet(A), H_\bullet(A, A_\sigma))$ forms a nc calculus again.

As a side remark, Nest-Tysgan suggested to consider $H^\bullet(A, A_\sigma)$ as a quantum analogue to the Fukaya category, while $H_\bullet(A, A_\sigma)$ was related by Kustermans, Murphy and Tuset to Woronowicz’s concept of covariant differential calculi over compact quantum groups.
The following is a slight generalisation of Ginzburg’s notion of a Calabi-Yau algebra (where $\sigma = \text{id}$):

**Definition**

An algebra $A$ is a **twisted Calabi-Yau algebra** with **modular automorphism** $\sigma \in \text{Aut}(A)$ if the $A^e$-module $A$ has (as an $A^e$-module) a finitely generated projective resolution of finite length and there exists $d \in \mathbb{N}$ and isomorphisms of right $A^e$-modules

$$\text{Ext}^i_{A^e}(A, A^e) \simeq \begin{cases} 0 & i \neq d, \\ A_\sigma & i = d. \end{cases}$$

The Ischebeck spectral sequence leads to a Poincaré-type duality

$$H^\bullet(A, A) \simeq H_{d-\bullet}(A, A_\sigma),$$

and by a result of Ginzburg/Lambre one shows that the BV module structure on the RHS yields a BV **algebra** structure (which we probably won’t discuss) on the LHS. Examples come from quantum groups, homogeneous spaces, the Podleś quantum sphere.
Example II: noncommutative Poisson structures

Define $O, M, o_i, e, 1, \bullet_j, t$ as in the example before, but replace the multiplication $\mu$ by a more general $\pi \in C^2(A)$ that fulfils

$$\pi \circ_1 \pi = \pi \circ_2 \pi.$$ 

In case $\pi$ is a cocycle, such a structure is called a noncommutative Poisson structure by Xu/Ginzburg, and can be seen as a translation of the property $[\pi, \pi]_{SN} = 0$ that defines a Poisson structure on a manifold.

The resulting simplicial differential $b = -\mathcal{L}_\pi$ is the algebraic Brylinski boundary that defines Poisson homology. Its original definition as

$$b^\pi := [\iota_\pi, d_{\text{deRham}},$$

is nothing than the Cartan-Rinehart homotopy. The resulting cosimplicial differential $\beta = \{\pi, \cdot\}$ is the algebraic Koszul-Lichnerowicz coboundary defining Poisson cohomology. $S_\varphi$ and $\mathcal{L}_\varphi$ look as before, but for $\smallfrown, \smile$ one obtains the new operators of Poisson cup and Poisson cap product:

$$(\varphi \smallfrown \psi)(a_1, \ldots, a_{p+q}) = \pi(\psi(a_1, \ldots, a_q), \varphi(a_{q+1}, \ldots, a_{p+q})),
\varphi \smile (a_0, \ldots, a_n) = (\pi(a_0, \varphi(a_1, \ldots, a_p)), a_{p+1}, \ldots, a_n),$$

Example III: calculi for left Hopf algebroids

The idea is to find a sort of bialgebra over a noncommutative base ring $A$. In a standard way, define $A$-rings (or $A$-algebras) and $A$-corings (or $A$-coalgebras) as monoids resp. comonoids in the category of $(A, A)$-bimodules. In particular, an $A^e$-ring $U$ can be described by two maps $s : A \to U$ and $t : A^{op} \to U$, called source and target.

Definition (Takeuchi, Lu, Xu)

A (left) $A$-bialgebroid is a $k$-module $U$ that is both an $A^e$-ring $(U, s, t)$ and an $A$-coring $(U, \Delta, \epsilon)$ such that the “coring-arising” $(A, A)$-bimodule structure coincides with that of the “ring-arising” left $A^e$-module structure, and such that

- $\Delta$ is a unital $k$-algebra morphism taking values in $U \times_A U$;
- $\epsilon(s(a)t(b)u) = a\epsilon(u)b$ and $\epsilon(uv) = \epsilon(us(\epsilon v)) = \epsilon(ut(\epsilon v))$.

Here, the Sweedler-Takeuchi subspace

$U \times_A U := \{ \sum_i u_i \otimes_A u_i' \in U \otimes_A U \mid \sum_i u_i t(a) \otimes_A u_i' = \sum_i u_i \otimes_A u_i' s(a) \}$

carries an algebra structure, in contrast to $U \otimes_A U$. 

Left Hopf algebroids

For a left bialgebroid \((U, A)\), define the (Hopf-)Galois map
\[
\beta : U \otimes_{A^{\text{op}}} U \rightarrow U \otimes_A U,
\quad u \otimes_{A^{\text{op}}} v \mapsto u(1) \otimes_A u(2)v.
\]

For bialgebras over fields, \(\beta\) is bijective if and only if \(U\) is a Hopf algebra, and \(\beta^{-1}(u \otimes_k v) := u(1) \otimes_k S(u(2))v\), where \(S\) is the antipode of \(U\). This motivates:

Definition (Schauenburg)

A left \(A\)-bialgebroid \(U\) is called a left Hopf algebroid (or \(\times_A\)-Hopf algebra) if \(\beta\) is a bijection.

We adopt a Sweedler-type notation
\[
u_{+} \otimes_{A^{\text{op}}} u_{-} := \beta^{-1}(u \otimes_A 1)
\]
for the so-called translation map \(\beta^{-1}(\cdot \otimes_A 1) : U \rightarrow U \otimes_{A^{\text{op}}} U\).
Examples of left Hopf algebroids

- For $A = k$, left Hopf algebroids are simply Hopf algebras.
- The enveloping algebra $A^e = A \otimes_k A^{\text{op}}$ of an arbitrary (unital) $k$-algebra $A$ is a left bialgebroid over $A$ by means of

$$s(a) := a \otimes_k 1, \quad t(b) := 1 \otimes_k b,$$

$$\Delta(a \otimes_k b) := (a \otimes_k 1) \otimes_A (1 \otimes_k b), \quad \epsilon(a \otimes_k b) := ab,$$

and a left Hopf algebroid by

$$(a \otimes_k b)_+ \otimes_{A^{\text{op}}} (a \otimes_k b)_- := (a \otimes_k 1) \otimes_{A^{\text{op}}} (b \otimes_k 1)$$

Observe: there is no notion of antipode for left Hopf algebroids.
Operads with multiplication from bialgebroids

Set

\[ C^\bullet(U, A) := \text{Hom}_{A^{\text{op}}} (U \otimes_{A^{\text{op}}} \cdot, A), \]

along with the structure maps

\[ \circ_i : C^p(U, A) \otimes C^q(U, A) \to C^{\mid p+q\mid}(U, A), \quad i = 1, \ldots, p, \]

where \( \mid p\mid := p - 1 \), given by

\[ (\varphi \circ_i \psi)(u^1, \ldots, u^{p+q-1}) := \varphi(u^1, \ldots, u^{i-1}, D\psi(u^i, \ldots, u^{i+q-1}), u^{i+q}, \ldots, u^{p+q-1}), \]

where \( D\psi(u^1, \ldots, u^q) := \psi(u^{(1)}, \ldots, u^{(1)}) \triangleright u^{(2)} \cdots u^{(q)} \), together with

\[ \mu := \varepsilon(m_U(\cdot, \cdot)), \quad 1 := \varepsilon(\cdot), \quad \text{and} \quad e := 1_A, \]

where \( m_U \) is the multiplication map of \( U \).
Gerstenhaber algebras from bialgebroids

**Theorem (K.-Krähmer, 2012)**

Let \((U, A)\) be a left bialgebroid. Then \(C^\bullet(U, A)\) with the structure maps given above forms an operad with multiplication.

Hence, by the theorem of Gerstenhaber-Schack/McClure-Smith mentioned before, one obtains:

**Corollary**

*The cohomology groups \(H^\bullet(U, A)\) (or \(\text{Ext}^\bullet_U(A, A)\) for a certain \(A\)-projectivity of \(U\) over \(A\)) form a Gerstenhaber algebra.*

For introducing *cyclic modules over operads* arising from bialgebroids, we need more structure:
Start with a right $U$-module $M$ that is simultaneously a left $U$-comodule; one needs to know what happens if action is followed by coaction: $M$ is an anti-Yetter-Drinfel’d module if

$$(mu)_{(−1)} \otimes_A (mu)_{(0)} = u − m_{(−1)} u_{+(1)} \otimes_A m_{(0)} u_{+(2)}$$

for all $u \in U$, $m \in M$ and stable if first coacting and then acting yields $m_{(0)} m_{(−1)} = m$,

(along with the fact that the induced $A$-module structures of action and coaction need to coincide).

The category $^U a\text{YD}_U$ of aYD modules is not monoidal but a module category over the category $^U \text{YD}_U$ of Yetter-Drinfel’d modules (to be introduced below), that is, tensoring them with a YD module gives an aYD again.
Moreover, to get a cyclic operad module, \((U, A)\) needs to be not only a bialgebroid but a left Hopf algebroid. For \(M \in ^{U}aYD_{U}\), we set

\[
C_{\bullet}(U, M) := M \otimes_{A^{op}} U^{\otimes A^{op}},
\]

and for the comp module maps

\[
\varphi \bullet_{i} (m, u^{1}, \ldots, u^{k}) := (m, u^{1}, \ldots, u^{i-1}, D_{\varphi}(u^{i}, \ldots, u^{i+p-1}), u^{i+p}, \ldots, u^{k}),
\]

for \(i = 1, \ldots, k - p + 1\), and any \(\varphi \in C^{p}(U, A)\), where \(p \geq 0\), and, as before,

\[
D_{\varphi}(u^{1}, \ldots, u^{p}) := \psi(u_{(1)}^{1}, \ldots, u_{(1)}^{p}) \triangleright u_{(2)}^{1} \cdots u_{(2)}^{p}.
\]
Theorem (K., 2013)

Let \((U, A)\) be a left Hopf algebroid and \(M\) an anti Yetter-Drinfel’d module. Then \(C_\bullet(U, M)\) forms a para-cyclic module over the operad \(C_\bullet(U, A)\) with multiplication w.r.t. the extra comp module map

\[
\varphi \bullet_0 (m, u^1, \ldots, u^k) := (m(0), u^1_+, \ldots, \varphi(u^{k-p+2}_+, \ldots, u^k_+, u^k_- \cdots u^1_- m(-1)) \triangleright u^{k-p+1}_+),
\]

where \(\varphi \in C^p(U, A)\), along with the cyclic operator

\[
t(m, u^1, \ldots, u^k) := (m(0) u^1_+, u^2_+, \ldots, u^k_+, u^k_- \cdots u^1_- m(-1)).
\]

If the anti Yetter-Drinfel’d module \(M\) is stable, then \(C_\bullet(U, M)\) is a cyclic unital module over the operad \(C_\bullet(U, A)\).
Hence, as said, one obtains a noncommutative differential calculus on the pair (cohomology, homology) and under certain projectivity assumptions on the pair \((\text{Ext}_U, \text{Tor}^U)\), which was proven before (K.-Krähmer, 2012).

**Question**

*Why is this interesting, in particular when looking at all these truly horrible formulae?*

- The answer is that many **classical homology theories** can be treated in a **unified manner** here:
  - in fact, inserting for the pair \((U, A)\) the pair \((A^e, A)\) yields the **Hochschild** example, but \((kG, k), (U(g), k), (V\Omega^1(X), C^\infty(X)), (VL, A)\) yield **group, Lie algebra, Poisson homology**, and **Lie algebroid homology**; also groupoids can be treated.
  - The **classical calculus structure from differential geometry** with Cartan’s homotopy formula \(\mathcal{L} = [d, \iota]\) is also contained.