"Symmetry" acting on observables acting on fields.

ELECTROMAGNETIC FIELD E, B

Invariants under the symmetries.

\[ N \xrightarrow{\text{VN-Act}} M \xrightarrow{\text{VN-Act}} \text{(soon will restrict to III-factors) \ldots} \]

Idea: The interplay between N and M should allow to recover the symmetries of (some kind of symmetries).

\[ \text{Group, Quantum Group, ...} \]

Example: \( N \xrightarrow{\text{VN-Act}} G \text{ Finite Group, } G \rightarrow \text{Out}(\text{N}) \)

\[ N \times G = N[G]\text{ Group Ring} \]

\[ N' \text{ M}_0 \quad N' \text{ M}_0 = CI \]

Question: Can we recognize G from the inclusion?
Given \( G \) acts on \( (C(G), \Delta) \), where \( \Delta \) is the coproduct, and \( \alpha \) is the coaction.

**Coaction of \( G \) on an Algebra \( A \):**

\[
\begin{align*}
\Delta: A &\rightarrow A \otimes C(G) \\
\text{id} \otimes \Delta: A \otimes C(G) \otimes C(G)
\end{align*}
\]

**Example:** \( G \) finite, \( C(G) \), \( \Delta: C(G) \rightarrow C(G \times G) \)

\[\Delta f(g, h) = f(gh)\]

**Suppose** \( G \to \text{Aut}(A) \)

\[
\Delta: A \rightarrow A \otimes C(G)
\]

**Dual:** \( (C^*(G), \hat{\Delta}) \)

**Example:** \( C^*(G) = \text{Group Algebra} \)

\[
\Delta: s_g \mapsto s_g \otimes s_g
\]

So \( G \mapsto (C(G), (\mu, \Delta)) \)

**Coaction** \( (C^*(G), (\hat{\mu}, \hat{\Delta})) \)

**Have that** \( C(G) \) acts on \( C^*(G) \):

\[
C(G) \rightarrow (C(G) \otimes C(G))
\]

\[
\text{(and coacts on itself)}
\]
Action of $C^*(G) \iff$ Coaction of $C(G)$

For a Hopf algebra, this is a definition of the action of the Hopf algebra on something, namely by a coaction of the dual coalgebra.

$A \overset{\delta}{\rightarrow} A \otimes C(G)$

$H \otimes \ell^2(G)$

$G$ (Quantum) Group

$A \times G := S(A) (\otimes C^*(G))$

(Acts on $H \otimes \ell^2(G)$)

A unitary $W : \ell^2(G) \otimes \ell^2(G) \otimes \ell^2(G)$

Computing with the arrow of $A$ on the first.

Two factors and "implement the coproduct on the $C^*(G)$ action, (?)

$l^2(G) \otimes l^2(G) = l^2(G, l^2(G))$

$W : f \rightarrow \lambda f$

$\langle W f, gh \rangle = f(g, g'h)$
\[ A \xrightarrow{\gamma} C^*(G) \]

\[ A \times G \rightarrow A \times G \otimes C^*(G) \quad \text{coaction} \]

Because \( G \) acts by outer automorphisms, \( \mathcal{N}' \cap M_0 = C_1 \) (centralizer of \( N \) inside \( M_0 \)).

Then

\[ N' \cap M_1 = C^*(\hat{G}) \]

\[ N' \cap M_2 = \overline{C(G) C^*(G)} = \mathcal{B}(l^2(G)) \]

\[ M_0' \cap M_2 = C^*(C^*(G)) \]

\[ \rightarrow \text{got } C^*(G) \text{ and } C^*(G) \text{ out of the sequence} \]