Exercise 8 (A, R, μ) Charmed TPF
⇒ (A-Mod, βR, θR) Ribbon Category

2) \((E \otimes 1)R = (1 \otimes e)R = 1\).

The Charmed Event: For \(U^{\text{char}}_{q-y}\).

Idea: Looking for invertible \(\mu \in U^{\text{char}}_{q-y}\).

Postulate: \(\mu = \prod K_i^{m_i}\). Use \(S^2(a) = \mu a \mu^{-1}\) to constrain \(\mu\).

Calculation:
\[
S^2(E_j) = \mu E_j \mu^{-1}, \quad S(E_j) = -E_j K_j^i
\]

\[
k_j E_j K_j^i = (\prod K_i^{m_i}) E_j (\prod K_i^{m_i})
\]

Using:
\[
SCE_j = -E_j K_j^i, \quad K_i E_j K_i^i = \delta_{ij} E_j
\]

\[
\Rightarrow S^2(E_j) = g \sum_{i,j} d_{i,j} \rightarrow \prod_{i,j} m_i d_{i,j} = \mu E_j \mu^{-1}
\]

Remark: \(a_{i,j} = \langle x_i | x_j \rangle \) → Need \(\langle x_i | x_j \rangle = \langle \Sigma m_i x_i | x_j \rangle \) for

Def: \(e \in \Lambda^*\) is the unique weight \(s.t.\ \langle x_i | x_j \rangle = \langle e | x_j \rangle\)

\[E_j = S \frac{\partial}{\partial x_j} \quad \frac{\partial}{\partial x_j} \quad \mu = \lambda\]
Conclusion: The exceptional element \( \mu = \Pi K^i_k \) where
\[ 2\mu = \sum m_k x_k. \]

By ribbon structure for \( sl_2 \):

Recall: Ribbon element is \( \mu = m(1 \otimes m) \mathbb{C}(\mathfrak{g}) \)
(Central, invertible)

Central \( \Rightarrow \) Acts by a scalar on \( \text{any rep } \mathfrak{g} \)
Determine that scalar by the action on a highest weight vector \( \lambda \)

\[ \mathbb{C}[m(1 \otimes m)] \lambda = \mathbb{C}[m(1 \otimes m)] (q^{\frac{1}{2}(\lambda \vee - \lambda)}) = q^{\lambda} \]

(\( \lambda \) highest weight, \( \mathfrak{g}^+ \)-raising \( \Rightarrow \) only \( t=0 \) term left)

\[ \lambda \in \text{highest weight} \]

\[ \lambda^2 = \lambda \cdot \lambda \]

(Recall: \( q^{\frac{1}{2}(\lambda \vee - \lambda)} \)): \( \lambda \times \lambda \rightarrow \mathbb{Z} \left( q^{\lambda^2} \right) \)

(\( \lambda, \mu \) \rightarrow \( q^{\frac{1}{2} H(\lambda) H(\mu)} \))

\[ \Theta_0 \in \text{Andrews' Handout} \]
The fusion category

Plan: Rep category of $U_q g$

$W_{kl}^{-1}$ simple category of tilting modules

$W_{kl}^{-1}$ reachable in $\text{Rep}_{\text{tilt}}$

$W_{kl}^{-1} \otimes W_{kl}^{-1} = \bigoplus W_{kl} \otimes W_{kl}$

This is semi-simple

$W_{kl}^{-1}$ is reachable in $\text{Rep}_{\text{tilt}}$

$W_{kl}^{-1} \otimes W_{kl}^{-1} \cong \bigoplus W_{kl} \otimes W_{kl} \otimes N_f$

Not semi-simple

Idea: Map on these $N_f$ to 0

$q^6 = 1$ \[ \Rightarrow \otimes \Rightarrow \Rightarrow = \Rightarrow \Rightarrow T_3 \]

Quantum dimension: $q_{\text{dim}} = \frac{m}{2}$

Here $q = \frac{1}{\sqrt{q_{\text{dim}}}}$

\[ q^{-3} q^{-3} \otimes q^{-3} q^{-3} \]

\[ \Rightarrow \downarrow \text{sum} \]

\[ \text{[4]} + [2] \]

\[ 1 + 1 = 0 \]

Wedge to mod out by some ideal of morphisms, in fact a tensor ideal. To make sure the quotient is still a tensor category. The idea should include $id_{T_3}$.
**Def:** A morphism \( \phi : V \rightarrow W \) is called **negligible** if 
\[ \text{tr}(\phi g) = 0 \quad \forall g : W \rightarrow V. \]

**Need:** \( \text{id}_{T \times \overline{l-1}} \) is negligible

**Eg:** \( q^6 = 1 \). Want that \( \forall f \in \text{End}(T_3) \), \( \text{tr}(f) = 0 \).

Already checked for \( \text{id} \). \( \langle \text{id}, n \rangle \)

For \( n \), \( \text{tr}(n) = \text{tr}(\mu^{-1} n) = 0 \).

**Def:** \( \text{Fus}_q \langle g \rangle = \text{Rep}_q \langle g \rangle \) **negligible morph**

\( \text{Hom}_\text{Fus}_q (V, W) = \text{Hom}_\text{Tilt, Neg(V, W)} \)

**Eg:** \( sl_2 \)

\( q^2 = 1, \overline{l} = 3 \)

Simple of Fus

\( \overline{l-1} \)

\( \text{Simple in Fus} \)

\( \overline{W} \)

\( \text{has } q \overline{d} = 0 \)

\( \text{killed} \)

\( \text{killed} \)
Need to check that the category satisfies the modularity condition.

Thm: $F_{u_1}^{\alpha_0} \varphi$ is modular if $D | l$ and

$$\frac{l}{D} \geq \langle \alpha_0, \varphi \rangle + 1$$

Highest root

$D =$
1. Type ADE
2. Type BCF
3. Type G

Example: $sl_2$, $l \geq 2$.

Ex 1: Exercise: Prove that $S_{na} \left\{ \binom{n}{m} \right\}_{m=0}^{l-2} = \left\{ (nt+1)(mt+1) \right\}_{n \in \mathbb{N}}$

Ex 2: Conclude that $F_{u_1}^{\alpha_0} sl_2$ is modular.