The ribbon structure on the quantum group

Balanced Hopf algebras and balanced tensor cat

Question: What structure on \((A/R)\) makes \((A,\text{mod},\beta)\) balanced?

Approach: \(A,\text{mod} \xrightarrow{\theta} A,\text{mod}

\begin{align*}
\forall M, \text{ need } M \\
\text{ try } I_{\text{mod}} \forall M
\end{align*}

Condition: 1) A-motive map: \(m \mapsto \theta_m \downarrow \downarrow \alpha_m \downarrow \downarrow \beta m \mapsto \beta_m \mapsto \beta m \Rightarrow \beta \text{ needs to be central}

2) Compatibility of unit and twist:

\[ \theta_1 = \text{id} \quad \text{(Defn. twist trivializes with the cop)} \]

\[ (m) \xrightarrow{\theta} \circ \xrightarrow{\theta_\circ} \circ \]

\[ m \mapsto \theta \cdot m = \epsilon(\alpha) m \]

(\text{action of A on } \circ \text{ via } \epsilon)

\[ \Rightarrow \text{ need } \epsilon(\alpha) = 1 \]

3) Compatibility of braiding and twist:

\[ \theta^2 = \Theta = \theta(\Theta \Theta) \]

\[ [m \text{ cop} \xrightarrow{R} \text{ mon} \xrightarrow{\beta} \text{ mon cop}] \]

\[ \beta m \mapsto \delta(\alpha) (\beta m \circ \beta m) \]
**Def**: A **braided Hopf algebra** $(A, R)$ is **balanced** given an invertible central element $v$ s.t. $(1)\overline{2}$.

**Ribbon Hopf Algebras and Ribbon Tensor Categories**

**Recall**: A **balanced tensor cat** is **ribbon** if

\[
\begin{align*}
A \cup v &= \mathbb{I} \\
v \cup A &= \mathbb{I}
\end{align*}
\]

(illustrated with an actual belt...)

**Twist**: $\alpha = 1$

**Calculation**: Given a **braided Hopf Alg** $(A, R)$, and $v \in A$-mod, then $v^* \otimes v$ is given by $v^* \otimes v$.

(Recall: $v = s(\delta v)$)

**Def**: Given a **braided Hopf Alg**, the **distinguished element** is $u := m(s^2) R$ (element occurring in $G_v$)

**Calculation**: $v^* \otimes v$ is given by...
Exercise *: \((S \otimes S)R = R\)

\[\begin{align*}
R & = R \\
\alpha & \in R \\
\delta & \in S(U)
\end{align*}\]

\textit{Distinguished Element}

\textbf{Def:} A \textit{braided Hopf algebra} \((A, R, \delta)\) is \textbf{ribbon} if:

1. \(S(\varphi) = \varphi\) \(\Rightarrow (\text{4})\)
2. \(S(\varphi) \varphi = \varphi^2 \Rightarrow (\text{3})\)

\textbf{Conclusion:} For a ribbon Hopf algebra \((A, R, \delta)\), \(A\)-mod is a ribbon tensor category.

\textbf{Not easy to give such a \(\varphi\) for our quantum group...}

\textbf{Charmed Hopf Algebra}

\textit{Idea:} Instead of directly defining \(\varphi\) \(\Rightarrow \varphi^2\), we'll suppose we have \(\varphi\) and define \(\varphi^2\).

\[\begin{align*}
\varphi \quad \text{and define} \quad \varphi^2
\end{align*}\]

\(\varphi^2\) is called the "charmed element"

\textbf{Def:} Given a \textit{braided Hopf algebra} \((A, R)\), an \textit{invertible element} \(\mu \in A\) is \textit{charmed} if:

1. \(\Delta(\mu) = \mu \otimes \mu\)
2. \(S(\mu) = \mu^{-1}\)
3. \(S^2(\mu) = \mu^{-1}\)
4. \(\mu(1 \otimes \mu^{-1}) R = \mu(1 \otimes \mu) \delta(R) \quad (= \varphi)\)
Comments: (3) \( \otimes \) = \text{left multiplication by } \mu \in A \text{ is an } A\text{-mod. map } V \rightarrow V^{**} \text{ for } V \text{ in } A\text{-mod.}

(4) \( \otimes \) = \text{the two above definitions of } \otimes \text{ agree} \quad (**) \quad (**) \quad (**) \quad (**) 

Note: 3) \( S^2(n) = n \quad \rightarrow \quad S(n^{-1}) = n^{-1} \text{ (exercise) } \quad V = U \quad (**) 

Prop: \((A, R, \mu)\) \text{ coquasitriangular, then } \((A, R, \mu)\) \text{ is ribbon.} 

\text{We wish we could just do that but will instead} 

Fact: \text{a ribbon Hopf algebra } \rightarrow \text{ coquasitriangular Hopf algebra} 

\text{? (probably, at least if } A \text{ f.d.)}