2-di TOFT'S: $O \rightarrow V$

Unit: $V \rightarrow (V \otimes V \rightarrow V)$

$[O \rightarrow V]

$V \rightarrow (V \otimes V \rightarrow V)$

Commutative

$F(S^1 \otimes S^1) = F(S^1) \otimes F(S^1)$

Twist:

$[a,b] \rightarrow [b,a]$

$F(S^2 \otimes S^1) = F(S^1) \otimes F(S^1)$

Coassociative, Coconmutative

Counital Coalgebra

Compatibility:

$\Rightarrow$ Commutative Frobenius Algebra
**Theorem:** \( \text{2dim-TOPT} \iff \text{COMMUTATIVE FROGENUS ALG} \)

**Example:**

**Fact:** Frob Alg = Algebra \( V \) with \( \langle \cdot, \cdot \rangle : V \otimes V \to C \)

Non-degenerate, s.t. \( \langle ab, c \rangle = \langle a, bc \rangle \)

\[ \langle \cdot, \cdot \rangle \Rightarrow X \]

Non-deg \( \Rightarrow \) exists \( E \)

Easiest way to write \( \langle a, b \rangle = E(a \otimes b) \)

\[ E \]

\[ \Rightarrow \text{Frob Alg = Unital Alg } V \text{ with } E : V \to C \]

s.t. \( \langle \cdot, \cdot \rangle = E \circ (\cdot, \cdot) \) is non-degenerate.

Easiest definition to get examples.

**Ex:** \( \mathcal{O} \), \( E(1) = \lambda \to 0 \). Then \( \Delta(1) = 1 \otimes 1 \otimes 1 \)

\[ C[t^Y]_{t^{+n}} \]

\( E(t^i) = \delta_{i,n} \)

\[ \Rightarrow \Delta(t^i) = t^i \otimes t^n + t^{i+n} \otimes t^{n-1} + \ldots + t^n \otimes t^i \]
Given a commutative Froeb algebra, want to construct a 2d-topf.

Use Morse theory.

Strategy:

1. \( \varepsilon = \sum_{M} - \) on \( H^*(M) \)

2. \( C \) to get an alg over \( C \)

\( \rightarrow \) gives a decomposition (by Morse theory) of any 2-di MFD as a composition of the above building blocks.

\( \Delta \) need the multiplication and co-multiplication to be (co)commutative for this to make sense.
Relations:

Given two Morse functions, $E$, a family of maps going from one to the other going through birth-death singularities:

![Diagrams]

To get a complete list: Write all the possible combinations of 2 critical points:
\[ \triangleleft \rightarrow \text{coassociativity} \]

\[ \circ \circ \sim [0 \circ \xrightarrow{m} \circ \xrightarrow{\Delta} 0] = [0 \circ \xrightarrow{m} \circ \xrightarrow{\Delta} 0] \]

is \( \Delta_0 \circ = \Delta_0 \circ \circ \) !

\[ \circ \rightarrow [0 \xrightarrow{\Delta} 0 \xrightarrow{m} \circ] = [0 \xrightarrow{\Delta} 0 \xrightarrow{m} \circ] \]

\[ \text{mod} \Delta = \text{mod} \Delta ! \]

\[ \rightarrow \text{get exactly all the relations of a Frobenius algebra} \]