Discrete Morse functions on infinite complexes

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joint work with
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Discrete, Computational and Algebraic Topology, Copenhagen 2014
Discrete Morse functions

A discrete Morse function $F$ on $M$ is a labelling of the cells with labels $t \in \mathbb{R}$ that is monotone increasing with respect to dimension, with at most one exception on each cell. That is, for each cell $\sigma(k)$ one of the following is true:

1. $F(\tau) \geq F(\sigma)$ for exactly one face $\tau(k-1) < \sigma$,
2. $F(\sigma) \geq F(\nu)$ for exactly one coface $\nu(k+1) > \sigma$,
3. $F(\tau) < F(\sigma) < F(\nu)$ for all faces $\tau$ and cofaces $\nu$.

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Cells that appear in $V$ are regular and cells that do not are critical.
Example 1

Here is an example of a discrete Morse function and the induced gradient vector field on a torus:
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A discrete Morse function on a finite cell complex has at least one critical vertex, at the minimal value.
Example 2

Here is an example on an infinite strip:
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On an infinite complex a discrete Morse function can have no critical cells.
Gradient paths

A sequence of adjoining arrows forms a gradient path or a $V$-path.
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A sequence of adjoining arrows forms a *gradient path* or a *V-path*.

This is a discrete vector field on a genus 2 surface with many *V*-paths.
A *discrete vector field* on $M$ is a partial pairing on the cells of $M$

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with every cell of $M$ in at most one pair of $W$. 

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A discrete vector field on a finite complex is the gradient field of a discrete Morse function if and only if it is acyclic (Forman).
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**Algorithm for assigning values:**
- on critical cells, the value is the dimension
- along $V$-paths of dimension $k$ assign decreasing values from $k$ towards $k - 1$, in case of a conflict (where $V$-paths merge) the lowest value wins.

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Morse inequalities on finite complexes

The following result is well known:

**Theorem (Forman)**

*If $F$ is a discrete Morse function on $M$ with $c_k$ critical cells of dimension $k$ and $b_k$ is the $k$-th Betti number of $M$, $k = 0, 1, \ldots, n$ (n is the dimension of $M$). Then:*

1. $c_k \geq b_k$ for all $k$,
2. $c_k - c_{k-1} + \cdots \pm c_0 \geq b_k - b_{k-1} + \cdots \pm b_0$, for all $k$,
3. $c_0 - c_1 + \cdots + (-1)^n c_n = b_0 - b_1 + \cdots + (-1)^n b_n = \chi(M)$.

Even more: the discrete gradient vector field gives an algorithm for computing the homology of $M$.

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Rays

$V$ a discrete vector field.

A $k$-ray in $V$ is an infinite sequence

$$\tau_0^{(k-1)} < \sigma_0^{(k)} > \tau_1^{(k-1)} < \sigma_1^{(k)}, \ldots$$

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Two rays are *equivalent* if they coincide from some common cell on.
Descending and ascending rays

A \textit{k-ray} is

\begin{itemize}
  \item \textit{descending} if the arrows go from $\tau_i$ to $\sigma_i$ for all $i$ and
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A descending ray is an infinite gradient path.
Critical elements

Theorem (Ayala, Fernández, Vilches)

If $F$ is a discrete Morse function on an infinite locally finite regular cell complex $M$ with $c_k$ critical cells and $d_k$ equivalence classes of descending rays, and if $b_k$ is the $k$-th Betti number of $M$, $k = 0, 1, \ldots, n$ ($n$ is the dimension of $M$). Then:

1. $(c_k + d_k) - (c_{k-1} + d_{k-1}) + \cdots \pm (c_0 + d_0) \geq b_k - b_{k-1} + \cdots \pm b_0$
   for all $k = 0, 1, \ldots, n - 1$,

2. $c_k + d_k \geq b_k$ for all $k = 0, 1, \ldots, n$,

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So, define the critical elements of a discrete Morse function on an infinite locally finite regular cell complex to be the critical cells and equivalence classes of maximal descending rays. $c_i$ and $d_i$ are all finite in the above theorem.
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Reversing rays

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The descending ray is replaced by the critical cell $\tau$ and an ascending ray.
Multirays

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Kukieła (in the more general context of Morse matchings on posets) defined the concept of a multiray, which is a ray along which there are infinitely many bypasses, and showed that multirays induces an infinite number of equivalence classes of rays.
Sublevel complexes

The sublevel complex of $F$ at $a$ is

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**Theorem (Forman’s Main theorem)**

*If $M$ is finite, then*

- *if $F^{-1}([a, b])$ contains no critical cells of $F$ then $M(b)$ collapses onto $M(a)$,*
- *if $F^{-1}([a, b])$ contains one critical cell of dimension $k$ then $M(b)$ is homotopy equivalent to $M(a)$ with a $k$-cell attached along its boundary.*
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In particular, $M$ has the homotopy type of a CW complex with one cell of dimension $k$ for each critical cell of $F$ of dimension $k$. 
Example

$M = M(2)$

$F$ has no critical cells, in particular no critical cell in $F^{-1}([1, 2])$, but $M(1)$ is not homotopy equivalent to $M(2)$. 

$M(1)$
Proper discrete Morse functions

The discrete Morse function $F$ is \textit{proper} if $F^{-1}([a, b])$ contains at most finitely many cells for any interval $[a, b]$. In this case Forman’s main theorem easily generalizes:

For a proper discrete Morse function Forman’s Main theorem is valid.
Existence of proper integrals

Which discrete vector fields have proper integrals?

No $V$-loops does not suffice:

For any interval $[-a, a]$, $F^{-1}([-a, a])$ is infinite.
Incident rays

The **descending region of a k-cell** $\sigma$ consists of all $k$ dimensional $V$-paths beginning in the boundary of $\sigma$. In addition, we add recursively all regular pairs $(\tau, \nu)$ of lower dimension in the boundary of their union with all cofaces of $\tau$ except $\nu$ already included.

The **descending region of a ray** is the union of the descending regions of its cells.

A ray $r_1$ of dimension $d_1$ is *incident* to a ray $r_2$ of dimension $d_2 > d_1$ when $D(r_1) \cap \overline{D(r_2)}$ contains infinitely many cells.
Classification theorem

A forbidden configuration is a descending ray with an incident ascending ray of lower dimension in the boundary of its descending region.

Theorem (Ayala, Jerše, M, Vilches)

A discrete vector field on a locally finite infinite regular cell complex $M$ with finitely many critical elements admits a proper integral if and only if it has no forbidden configurations.

On complexes of dimension 1 (graphs) no $V$-loops suffices.
Proof

The proof is an algorithm for constructing such an integral:

1. all critical cells are given the value equal to their dimension,
2. $M$ is expressed as the union $M = \bigcup_{i=0}^{\infty} K_i$ of an increasing sequence of finite sub complexes with the property, that each $K_i$ intersects any ray in only one component, and that it includes the whole closed descending region of a cell that does not belong to any ray,
3. $F$ is defined on $K_0$ essentially by carefully assigning decreasing values along $V$-paths,
4. $F$ is inductively extended from $K_i$ to $K_{i+1}$ in the same way.
Homology

Theorem (Kukieła)

If $M$ is a regular cell complex (not necessarily locally finite) with an acyclic discrete gradient field $V$ that has a finite number of equivalence classes of maximal descending rays then there exists an acyclic gradient field $V'$ with no descending rays, with one critical cell of dimension $k$ for each critical cell of $V$ of the same dimension and in addition a critical cell of dimension $k$ for each descending $k$-ray of $V$. 

This discrete vector field gives a Morse chain complex with homology isomorphic to the homology of $K$. Actually the theorem is proved in the more general setting of matchings on posets.
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At the very end: an attempt at motivation

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It follows from the Stallings-Swan theorem its cohomological dimension is 2, so there exists a nontrivial 2-cocycle $a \in H^2(\pi; M)$ for some $\pi$-module $M$. 
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The Berstein class $b$ is an element of $H^1(G; I)$, where $I = \ker \mathbb{Z}[G] \rightarrow \mathbb{Z}$ is the augmentation ideal.
The grope (by Aleš Vavpetič)

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Could we obtain a nontrivial dimension two cohomology element from (some version of) discrete Morse theory?

Thank you!