Covering dimension using toric varieties

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Recall a couple of classical theorems:

Theorem (Lebesgue)

If the unit cube $[0,1]^n$ is covered by closed sets so that no point is covered more than n times then one of the covering sets must intersect two opposite facets of the cube.

Theorem (Knaster–Kuratowski–Mazurkiewicz)

If a simplex $\Delta^n \subset \mathbb{R}^n$ is covered by closed sets so that no point is covered more than n times then one of the covering sets intersects all the facets of Δ^n.

These theorems allow to justify the notion of covering dimension for the standard space $\mathbb{R}^n$; they are just basic facts of the theory of covering dimension.

Also, they are very popular in mathematical economics: KKM lemma serves as a usual tool to establish the existence of equilibria in certain economical situations (games); sometimes it even allows to find those equilibria efficiently.

Different proofs of these theorems are known; they are not so hard in fact.

The most popular way is to subdivide the cube or the simplex into a triangulation and establish some combinatorial statement, like Sperner's lemma on coloring the vertices of a triangulation of a simplex; or the HEX lemma for coloring the triangulation of a cube.

We want to exploit a different approach and apply some elementary techniques of toric geometry to prove these theorems.

These proofs may seem harder and maybe less straightforward. Still, we explore this connection to toric geometry and eventually prove some generalizations of the classical theorems.

In order to work with the KKM-type theorems, we consider the moment map

$$\pi : \mathbb{C}P^n \mapsto \Delta^n,$$

where

$$\pi(z_0 : z_1 : \dots : z_n) = \frac{(|z_0|^2, \dots, |z_n|^2)}{|z_0|^2 + \dots + |z_n^2|}.$$

When considering $\mathbb{C}P^n$ (and any other smooth algebraic variety M), we are mostly interested in its Kähler form $\omega \in \Omega^{1,1}(M)$. This is a closed 2-form, its power ω^n is the volume form, and therefore the cohomology class $[\omega^n] \in H^{2n}(M; \mathbb{R})$ does not vanish.

We next observe the following property of the cohomology multiplication (due to Lyusternik and Schnirelmann): If $U_1, \dots, U_m$ are open subsets of M such that the cohomology class $[\omega]$ vanishes on every U_i (we call such subsets of M inessential) then the class $[\omega^m]$ vanishes on their union $U_1 \cup \dots \cup U_m$.

An easy way to note this is to find 1-forms α_i defined on the whole M such that $\omega = d\alpha_i$ on its respective U_i and then note that the product

$$(\omega - d\alpha_1) \wedge \dots \wedge (\omega - d\alpha_m)$$

vanishes on the union $U_1 \cup \dots \cup U_m$. Then expanding this expression it is easy to express ω^m as the differential of some other form.

We will also use the following crucial lemma:

Lemma (Palais)

An open covering $\mathcal{U}$ of a paracompact space with multiplicity at most m can be refined to an m-colorable open covering, that is a covering consisting of m subfamilies $\mathcal{U}_1, \dots, \mathcal{U}_m$ (colors) such that for any sets U_1, U_2 in the same color class $\mathcal{U}_i$ the intersection $U_1 \cap U_2$ is empty.

Informally, it corresponds to the barycentric subdivision of the nerve of a covering.

If we are interested in open coverings $\mathcal{U}$ of (an algebraic variety) M by inessential sets with multiplicity at most m, then we pass to a colored refinement and then take the sets $U_i^* = \bigcup \mathcal{U}_i$. These sets turn out to be inessential and we finally conclude:

Proposition

If the family $\mathcal{U}$ of open subsets of M consists of inessential open sets and has multiplicity at most m then $[\omega^m]$ vanishes on the union $\bigcup \mathcal{U}$.

This is our main tool in what follows. Also, we will use the same proposition for closed coverings, this is possible by standard compactness argument.

So how the KKM theorem is now deduced?

Let a closed covering $\mathcal{V}$ of Δ^n correspond to a closed covering $\mathcal{U}$ of $\mathbb{C}P^n$. Assume the contrary: Every $V \in \mathcal{V}$ has empty intersection with some facet F_i of Δ^n. Then its corresponding $U \in \mathcal{U}$ has no intersection with the corresponding hyperplane $z_i = 0$.

We know from the elementary algebraic geometry that the cohomological class of ω is Poincaré dual to any hypeplane in the projective space M; and therefore can be realized by a differential form ω' supported in an arbitrarily small neighborhood of $H_i = \{z_i = 0\}$. So U is inessential and the covering of the whole $\mathbb{C}P^n$ by inessential sets must have multiplicity at least $n + 1$. The proof is complete.

For Lebesgue's theorem, the argument is similar, with the following differences. The moment map now is $\pi : (\mathbb{C}P^1)^{\times n} \to Q^n$, and the hyperplane divisor can be realized as the combination $\pi^{-1}(F_1) + \dots + \pi^{-1}(F_n)$, where $\{F_i\}$ is a set of some n facets of the cube having a common vertex.

If a covering set $V \subseteq Q^n$ does not touch any pair of opposite facets then it has no intersection with some such collection $\{F_i\}_{i=1}^n$, and therefore $U = \pi^{-1}(V)$ is inessential. So Lebesgue's theorem follows similarly.

This is not a big deal to give such tricky proofs for those old and easy theorems.

So we list several generalizations that arise naturally in this approach.

Theorem

Let $\{V_i\}$ be a family of closed subsets of Δ^n, none of them touching all facets of Δ^n. If this family $\{V_i\}$ has covering multiplicity at most k then there exists a connected component of the complement $\Delta^n \setminus \bigcup_i V_i$ that intersects every k-face of Δ^n.

This result was conjectured by Dömötör Pálvölgyi. The proof is the same as above, we only have to notice that ω^{n-k} does not vanish on $\pi^{-1}(\Delta^n \setminus \bigcup_i V_i)$, and hence does not vanish on some its connected component. Since ω^{n-k} is Poincaré dual to the preimage of every k-face of Δ^n, the result follows.

The corresponding generalization of Lebesgue's theorem is:

Theorem

Let $\{V_i\}$ be a family of closed subsets of the unit cube Q^n such that none of V_i touches a pair of opposite facets of Q^n. If the covering multiplicity of $\{V_i\}$ is at most k then there exists a connected component Z of the complement $Q^n \setminus \bigcup_i V_i$ and a k-dimensional coordinate subspace $L \subseteq \mathbb{R}^n$ such that Z intersects every k-face of Q^n parallel to L.

The proof is similar to the previous one.

As another example we give a unified statement of the KKM and Lebesgue theorems:

Theorem

Let a simple polytope $P \subset \mathbb{R}^n$ be covered by closed sets with covering multiplicity at most n. Then one of the covering sets touches at least $n+1$ facets of P.

In this case the toric variety may be non-smooth, but it is normal and it is known that we can work with hyperplane divisors similarly to the smooth case.

The following theorem can also be proved within this toric approach:

Theorem (The topological central point theorem)

Let $m = (d+1)k$ and let W be a d-dimensional metric space. Suppose $f : \Delta^m \to W$ is a continuous map. Then

$$\bigcap_{\substack{F \subset \Delta^m \\ \dim F = dk}} f(F) \neq \emptyset,$$

where the intersection is taken over all faces of Δ^m of dimension dk.

This theorem for linear maps of Δ^m into $\mathbb{R}^d$ becomes the well known central point theorem.

Sketch of the proof:

For a sufficiently fine open covering $\{W_i\}$ of W of multiplicity at most $d+1$ we consider the corresponding covering of Δ^m by $V_i = f^{-1}(W_i)$, and the covering of $\mathbb{C}P^m$ by $U_i = \pi^{-1}(V_i)$. Similar to the above proofs, from non-vanishing of $[\omega]^m$ over the entire $\mathbb{C}P^m$ we conclude that for some U_i the class $[\omega]^k$ does not vanish over U_i.

Hence, by the argument that we have already used, the corresponding V_i must intersect all faces of Δ^m of codimension k, that is of dimension dk. Then the standard compactness argument establishes the same for a preimage $f^{-1}(y)$ of some point $y \in W$, to which the sequence of chosen W_i's tend.

Previously, the author gave another proof of the topological central point theorem that can be interpreted now as a real toric version of the given complex toric argument.

Thank you!