Homology of Moduli Spaces of Riemann Surfaces

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Hilbert Unification

Let $\mathcal{MS}_g = \mathcal{MS}_g^{\text{par}} \cup \mathcal{MS}_g^{\text{nonpar}}$ be the moduli spaces of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ boundary curves and $m \geq 0$ punctures resp. with $n \geq 1$ incoming and $m \geq 1$ outgoing boundary curves. At the beginning of our studies is a uniformization method (called Hilbert unification), which is useful in the following aspects:

1. It allows the determination of several operad structures and thus of homology operations.
2. It gives a cell decomposition.
3. Which in turn leads to the computation of homology groups.

We replace (for technical reasons) in the parallel case $\mathcal{MS}_g^{\text{par}}$ each boundary curve by a point $Q$ with a non-zero tangent vector $T$. For the Hilbert unification we look at all harmonic functions $u: \mathcal{F} \to \mathcal{R}$ $u: \mathcal{R}$ with the following singularities: In the case $\mathcal{MS}_g^{\text{par}}$ we require a simple singularity at the $n$ points $Q_1, \ldots, Q_n$ with directions $T_1, \ldots, T_n$, and logarithmic singularities at the punctures $P_1, \ldots, P_m$ in the case $\mathcal{MS}_g^{\text{nonpar}}$ we require $u$ to be positive and constant on the incoming curves and vanishing on the outgoing curves. Such a function $u$ is uniquely determined by choosing the constant together with integration constants for $u$ and for the harmonic conjugates $\overline{u}$ (defined only as a differential) these constants determine a point in a vector bundle $\mathcal{B}^{\text{par}} \to \mathcal{MS}_g^{\text{par}} \cup \mathcal{MS}_g^{\text{nonpar}}$. We cut the surface $F$ along the critical graph (i.e. all flow lines of the gradient flow of $u$) and $u$ then determines a contractible subgraph resp. $n$ annular regions (parallel case) resp. $m$ annular regions (radial case). We can map them biholomorphically onto $n$ planes with parallel slits resp. onto annuli with radial slits. In the following figures below we picture the parallel case with $q = 1, n = 1, m = 1$. The critical graph is drawn in red.

The slit pictures are determined by the combinatorics of their gluing: permutations in the symmetric group $\mathcal{S}_n$, where $0 \leq p \leq 2n$ is the number of critical levels of $u$ and $0 \leq q \leq 2n$ the number of critical levels of $v$. Here $0 = 2g - 2 + m + 2n$ in the parallel case resp. $0 = 2g - 2 + m + n$ in the radial case. The distances of the critical levels of $u$ resp. $v$ give two sets of barycentric coordinates. Thus we obtain a finite bi-simplicial complex $\mathcal{P} \to \mathcal{P}$ resp. $\mathcal{P} \to \mathcal{P}$ of simplicial complexes with $\mathcal{P}$ containing “degenerate” subcomplexes; then the Hilbert unification is a homomorphism $H_{\text{par}}(\mathcal{R}, \mathcal{R}) \to \mathcal{P}$.

If $\mathcal{P}$ denotes the orientation system, we can compute the homology via Poincaré duality $H_{\text{par}}(\mathcal{R}, \mathcal{R}) \cong H_{\mathcal{P}}(\mathcal{P}, \mathcal{P}) \cong \mathcal{P}$. In: Homologieberechnungen von Modulräumen Riemannscher Flächen durch diskrete Methoden, 2008.

Homology Operations. Staring at these slit pictures (parallel or radial) it is apparent how to define a multiplication (i.e., a connected sum operation), yet even various operators, not only the little-$2$-cube operator, acting on these moduli spaces. Thus there is Hopf algebra structure, Dy-Lashed operations and many more operations on the homology of moduli spaces.

Small chain complexes. As seen above, the homology of the moduli space can be computed as the (co)homology of the associated simplicial complex $F = F/p$ resp. $F = F$. The homology of this bi-complex has a remarkable property, noticed by Ehrenfried in [Ehr], its homology is concentrated in the top degree $q = h$. The reason for this, found by Vay in [Vay], is the factorability of the symmetric groups: this property, shared by many Coxeter groups, gives the same homology as the chain complex much smaller than the bar resolution computing the homology of groups. Our calculations could therefore be reduced to a manageable chain complex $\mathcal{S}$ which we call Ehrenfried complex. The following two tables demonstrate this reduction.

Calculations

The cluster spectral sequence computing the rational homology of $\mathcal{MS}_g(1, 1)$ is as follows.

| $\mathcal{S}_n \to \mathcal{H}_n(\mathcal{MS}_g(1, 1))$ |
|------------------|------------------|
| $\mathcal{S}_n$   | $\mathcal{H}_n(\mathcal{MS}_g(1, 1))$ |
| $\mathcal{S}_n$   | $\mathcal{H}_n(\mathcal{MS}_g(1, 1))$ |
| $\mathcal{S}_n$   | $\mathcal{H}_n(\mathcal{MS}_g(1, 1))$ |

Using a well-known calculation for $H_{\text{nonpar}}(\mathcal{S}_n, \mathcal{S}_n)$ of Powell [Pow] and one for $H_{\text{par}}(\mathcal{S}_n, \mathcal{S}_n)$ by Sushkov [Sush], we state here a few examples of our calculations.

The homology of the moduli spaces $\mathcal{MS}_g$ and $\mathcal{MS}_g(1, 1)$ is

The integral homology of the moduli space $\mathcal{M}_{g, 1}$ is

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References


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