HOMOLOGICAL ALGEBRA IN BIVARIANT K-THEORY AND OTHER TRIANGULATED CATEGORIES. I

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Abstract. Bivariant (equivariant) K-theory is the standard setting for non-commutative topology. We may carry over various techniques from homotopy theory and homological algebra to this setting. Here we do this for some basic notions from homological algebra: phantom maps, exact chain complexes, projective resolutions, and derived functors. We introduce these notions and apply them to examples from bivariant K-theory.

An important observation of Beligiannis is that we can approximate our category by an Abelian category in a canonical way, such that our homological concepts reduce to the corresponding ones in this Abelian category. We compute this Abelian approximation in several interesting examples, where it turns out to be very concrete and tractable.

The derived functors comprise the second page of a spectral sequence that, in favourable cases, converges towards Kasparov groups and other interesting objects. This mechanism is the common basis for many different spectral sequences. Here we only discuss the very simple 1-dimensional case, where the spectral sequences reduce to short exact sequences.

1. Introduction

It is well-known that many basic constructions from homotopy theory extend to categories of C*-algebras. As we argued in [17], the framework of triangulated categories is ideal for this purpose. The notion of triangulated category was introduced by Jean-Louis Verdier to formalise the properties of the derived category of an Abelian category. Stable homotopy theory provides further classical examples of triangulated categories. The triangulated category structure encodes basic information about manipulations with long exact sequences and (total) derived functors. The main point of [17] is that the domain of the Baum-Connes assembly map is the total left derived functor of the functor that maps a G-C*-algebra A to K_\ast (G \ltimes_r A).

The relevant triangulated categories in non-commutative topology come from Kasparov’s bivariant K-theory. This bivariant version of K-theory carries a composition product that turns it into a category. The formal properties of this and related categories are surveyed in [15], with an audience of homotopy theorists in mind.

Projective resolutions are among the most fundamental concepts in homological algebra; several others like derived functors are based on it. Projective resolutions seem to live in the underlying Abelian category and not in its derived category. This is why total derived functors make more sense in triangulated categories than the derived functors themselves. Nevertheless, we can define derived functors in triangulated categories and far more general categories. This goes back to Samuel

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Eilenberg and John C. Moore [9]. We learned about this theory in articles by Apostolos Beligiannis [4] and J. Daniel Christensen [8].

Homological algebra in non-Abelian categories is always relative, that is, we need additional structure to get started. This is useful because we may fit the additional data to our needs. In a triangulated category $\mathfrak{T}$, there are several kinds of additional data that yield equivalent theories; following [8], we use an ideal in $\mathfrak{T}$. We only consider ideals $\mathcal{I}$ in $\mathfrak{T}$ of the form

$$\mathcal{I}(A, B) := \{ x \in \mathfrak{T}(A, B) \mid F(x) = 0 \}$$

for a stable homological functor $F: \mathfrak{T} \to \mathfrak{C}$ into a stable Abelian category $\mathfrak{C}$. Here stable means that $\mathfrak{C}$ carries a suspension automorphism and that $F$ intertwines the suspension automorphisms on $\mathfrak{T}$ and $\mathfrak{C}$, and homological means that exact triangles yield long exact sequences. Ideals of this form are called homological ideals.

A basic example is the ideal in the Kasparov category $\mathcal{KK}$ defined by

$$(1.1) \quad \mathcal{I}_K(A, B) := \{ f \in \mathcal{KK}(A, B) \mid 0 = K_*(f) : K_*(A) \to K_*(B) \}. $$

For a compact (quantum) group $G$, we define two ideals $\mathcal{I}_n, \mathcal{I}_{n, K}$ in the equivariant Kasparov category $\mathcal{KK}^G$ by

$$(1.2) \quad \mathcal{I}_n(A, B) := \{ f \in \mathcal{KK}^G(A, B) \mid G \ltimes f = 0 \text{ in } \mathcal{KK}(G \ltimes A, G \ltimes B) \},$$

$$\mathcal{I}_{n, K}(A, B) := \{ f \in \mathcal{KK}^G(A, B) \mid K_*(G \ltimes f) = 0 \},$$

where $K_*(G \ltimes f)$ denotes the map $K_*(G \ltimes A) \to K_*(G \ltimes B)$ induced by $f$.

For a locally compact group $G$ and a (suitable) family of subgroups $\mathcal{F}$, we define the homological ideal

$$(1.3) \quad \mathcal{V}C_{\mathcal{F}}(A, B) := \{ f \in \mathcal{KK}^G(A, B) \mid \text{Res}^H_G(f) = 0 \text{ in } \mathcal{KK}^H(A, B) \text{ for all } H \in \mathcal{F} \}. $$

If $\mathcal{F}$ is the family of compact subgroups, then $\mathcal{V}C_{\mathcal{F}}$ is related to the Baum–Connes assembly map ([17]). Of course, there are analogous ideals in more classical categories of (spectra of) $G$-CW-complexes.

All these examples can be analysed using the machinery we explain. We carry this out in some cases in Sections [4] and [3].

We use an ideal $\mathcal{I}$ to carry over various notions from homological algebra to our triangulated category $\mathfrak{T}$. In order to see what they mean in examples, we characterise them using a stable homological functor $F: \mathfrak{T} \to \mathfrak{C}$ with ker $F = \mathcal{I}$. This is often easy. For instance, a chain complex with entries in $\mathfrak{T}$ is $\mathcal{I}$-exact if and only if $F$ maps it to an exact chain complex in the Abelian category $\mathfrak{C}$ (see Lemma [28], and a morphism in $\mathfrak{T}$ is an $\mathcal{I}$-epimorphism if and only if $F$ maps it to an epimorphism. Here we may take any functor $F$ with ker $F = \mathcal{I}$.

But the most crucial notions like projective objects and resolutions require a more careful choice of the functor $F$. Here we need the universal $\mathcal{I}$-exact functor, which is a stable homological functor $F$ with ker $F = \mathcal{I}$ such that any other such functor factors uniquely through $F$ (up to natural equivalence). The universal $\mathcal{I}$-exact functor and its applications are due to Apostolos Beligiannis [4].

If $F: \mathfrak{T} \to \mathfrak{C}$ is universal, then $F$ detects $\mathcal{I}$-projective objects, and it identifies $\mathcal{I}$-derived functors with derived functors in the Abelian category $\mathfrak{C}$ (see Theorem [59]). Thus all our homological notions reduce to their counterparts in the Abelian category $\mathfrak{C}$.
In order to apply this, we need to know when a functor $F$ with $\ker F = \mathcal{I}$ is the universal one. We develop a new, useful criterion for this purpose here, which uses partially defined adjoint functors (Theorem 57).

Our criterion shows that the universal $\mathcal{I}_K$-exact functor for the ideal $\mathcal{I}_K$ in $\text{KK}$ in (1.1) is the K-theory functor $K_*$, considered as a functor from $\text{KK}$ to the category $\text{Ab}_{\mathbb{Z}/2}$ of countable $\mathbb{Z}/2$-graded Abelian groups (see Theorem 63). Hence the derived functors for $\mathcal{I}_K$ only involve Ext and Tor for Abelian groups.

For the ideal $\mathcal{I}_n$ in $\text{KK}_G$ in (1.3), we get the functor

$$\text{KK}^G \to \text{Mod}(\text{Rep} G)_{\mathbb{Z}/2}^c, \quad A \mapsto K_*(G \rtimes A),$$

where $\text{Mod}(\text{Rep} G)_{\mathbb{Z}/2}^c$ denotes the Abelian category of countable $\mathbb{Z}/2$-graded modules over the representation ring $\text{Rep} G$ of the compact (quantum) group $G$ (see Theorem 72). Here we use a certain canonical $\text{Rep} G$-module structure on $K_*(G \rtimes A)$. Hence derived functors with respect to $\mathcal{I}_n$ involve Ext and Tor for $\text{Rep} G$-modules.

We do not need the $\text{Rep} G$-module structure on $K_*(G \rtimes A)$ to define $\mathcal{I}_n$: our machinery notices automatically that such a module structure is missing. The universality of the functor in (1.5) clarifies in what sense homological algebra with $\text{Rep} G$-modules is a linearisation of algebraic topology with $G$-$C^*$-algebras.

The universal homological functor for the ideal $\mathcal{I}_n$ is quite similar to the one for $\mathcal{I}_K$ (see Theorem 73). There is a canonical $\text{Rep} G$-module structure on $G \rtimes A$ as an object of $\text{KK}$, and the universal $\mathcal{I}_n$-exact functor is essentially the functor $A \mapsto G \rtimes A$, viewed as an object of a suitable Abelian category that encodes this $\text{Rep} G$-module structure; it also involves a fully faithful embedding of $\text{KK}$ in an Abelian category due to Peter Freyd ([10]).

The derived functors that we have discussed above appear in a spectral sequence which – in favourable cases – computes morphism spaces in $\mathfrak{S}$ (like $\text{KK}^G(A, B)$) and other homological functors. This spectral sequence is a generalisation of the Adams spectral sequence in stable homotopy theory and is the main motivation for [8]. Much earlier, such spectral sequences were studied by Hans-Berndt Brinkmann in [7]. In [16], this spectral sequence is applied to our bivariant K-theory examples. Here we only consider the much easier case where this spectral sequence degenerates to an exact sequence (see Theorem 66). This generalises the familiar Universal Coefficient Theorem for $\text{KK}_*(A, B)$.

2. Homological ideals in triangulated categories

After fixing some basic notation, we introduce several interesting ideals in bivariant Kasparov categories; we are going to discuss these ideals throughout this article. Then we briefly recall what a triangulated category is and introduce homological ideals. Before we begin, we should point out that the choice of ideal is important because all our homological notions depend on it. It seems to be a matter of experimentation and experience to find the right ideal for a given purpose.

2.1. Generalities about ideals in additive categories. All categories we consider will be additive, that is, they have a zero object and finite direct products and coproducts which agree, and the morphism spaces carry Abelian group structures such that the composition is additive in each variable ([13]).

**Notation 1.** Let $\mathcal{C}$ be an additive category. We write $\mathcal{C}(A, B)$ for the group of morphisms $A \to B$ in $\mathcal{C}$, and $A \in \mathcal{C}$ to denote that $A$ is an object of the category $\mathcal{C}$.
Definition 2. An ideal $\mathcal{I}$ in $\mathcal{C}$ is a family of subgroups $\mathcal{I}(A, B) \subseteq \mathcal{C}(A, B)$ for all $A, B \in \mathcal{C}$ such that

$$\mathcal{C}(C, D) \cap \mathcal{I}(B, C) \cap \mathcal{C}(A, B) \subseteq \mathcal{I}(A, D)$$

for all $A, B, C, D \in \mathcal{C}$.

We write $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if $\mathcal{I}_1(A, B) \subseteq \mathcal{I}_2(A, B)$ for all $A, B \in \mathcal{C}$. Clearly, the ideals in $\mathcal{T}$ form a complete lattice. The largest ideal $\mathcal{C}$ consists of all morphisms in $\mathcal{C}$; the smallest ideal $\mathcal{O}$ contains only zero morphisms.

Definition 3. Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories and let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. Its kernel $\ker F$ is the ideal in $\mathcal{C}$ defined by

$$\ker F(A, B) := \{ f \in \mathcal{C}(A, B) \mid F(f) = 0 \}.$$

This should be distinguished from the kernel on objects, consisting of all objects with $F(A) \cong 0$, which is used much more frequently. The kernel on objects is the class of $\ker F$-contractible objects that we introduce below.

Definition 4. Let $\mathcal{I} \subseteq \mathcal{T}$ be an ideal. Its quotient category $\mathcal{C}/\mathcal{I}$ has the same objects as $\mathcal{C}$ and morphism groups $\mathcal{C}(A, B)/\mathcal{I}(A, B)$.

The quotient category is again additive, and the obvious functor $F : \mathcal{C} \to \mathcal{C}/\mathcal{I}$ is additive and satisfies $\ker F = \mathcal{I}$. Thus any ideal $\mathcal{I}$ in $\mathcal{C}$ is of the form $\ker F$ for a canonical additive functor $F$.

The additivity of $\mathcal{C}/\mathcal{I}$ and $F$ depends on the fact that any ideal $\mathcal{I}$ is compatible with finite products in the following sense: the natural isomorphisms

$$\mathcal{C}(A, B_1 \times B_2) \cong \mathcal{C}(A, B_1) \times \mathcal{C}(A, B_2), \quad \mathcal{C}(A_1 \times A_2, B) \cong \mathcal{C}(A_1, B) \times \mathcal{C}(A_2, B)$$

restrict to isomorphisms

$$\mathcal{I}(A, B_1 \times B_2) \cong \mathcal{I}(A, B_1) \times \mathcal{I}(A, B_2), \quad \mathcal{I}(A_1 \times A_2, B) \cong \mathcal{I}(A_1, B) \times \mathcal{I}(A_2, B).$$

2.2. Examples of ideals.

Example 5. Let $KK$ be the Kasparov category, whose objects are the separable $C^*$-algebras and whose morphism spaces are the Kasparov groups $KK_0(A, B)$, with the Kasparov product as composition. Let $\mathfrak{Ab}^{\mathbb{Z}/2}$ be the category of $\mathbb{Z}/2$-graded Abelian groups. Both categories are evidently additive.

K-theory is an additive functor $K_* : KK \to \mathfrak{Ab}^{\mathbb{Z}/2}$. We let $\mathcal{I}_K := \ker K_*$ (as in (1.1)). Thus $\mathcal{I}_K(A, B) \subseteq KK(A, B)$ is the kernel of the natural map

$$\gamma : KK(A, B) \to \text{Hom}(K_*(A), K_*(B)) := \prod_{n \in \mathbb{Z}/2} \text{Hom}(K_n(A), K_n(B)).$$

There is another interesting ideal in KK, namely, the kernel of a natural map

$$\kappa : \mathcal{I}_K(A, B) \to \text{Ext}(K_*(A), K_{*+1}(B)) := \prod_{n \in \mathbb{Z}/2} \text{Ext}(K_n(A), K_{n+1}(B))$$

due to Lawrence Brown (see [23]), whose definition we now recall. We represent $f \in KK(A, B) \cong \text{Ext}(A, C_0(\mathbb{R}, B))$ by a $C^*$-algebra extension $C_0(\mathbb{R}, B) \otimes \mathbb{K} \to E \to A$. This yields an exact sequence

$$\begin{array}{cccccc}
K_1(B) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) & \longrightarrow & 0 \\
\downarrow f_* & & & \downarrow & f_* & & \\
K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_0(B). & \end{array}$$
The vertical maps in (2.1) are the two components of \( \gamma(f) \). If \( f \in \mathcal{I}_K(A, B) \), then (2.1) splits into two extensions of Abelian groups, which yield an element \( \kappa(f) \) in \( \text{Ext}((K_*(A), K_{*-1}(B))) \).

**Example 6.** Let \( G \) be a second countable, locally compact group. Let \( \mathcal{IC}^G \) be the associated equivariant Kasparov category; its objects are the separable \( G \)-C*-algebras and its morphism spaces are the groups \( \mathcal{IC}^G(A, B) \), with the Kasparov product as composition. If \( H \subseteq G \) is a closed subgroup, then there is a restriction functor \( \text{Res}^H_H : \mathcal{IC}^H \to \mathcal{IC}^H \), which simply forgets part of the equivariance.

If \( \mathcal{F} \) is a set of closed subgroups of \( G \), we define an ideal \( \mathcal{VC}_\mathcal{F} \) in \( \mathcal{IC}^G \) by

\[
\mathcal{VC}_\mathcal{F}(A, B) := \{ f \in \mathcal{IC}^G(A, B) \mid \text{Res}^H_H(f) = 0 \quad \text{for all} \quad H \in \mathcal{F} \}
\]

as in (1.4). Of course, the condition \( \text{Res}^H_H(f) = 0 \) is supposed to hold in \( \mathcal{IC}^H(A, B) \). We are mainly interested in the case where \( \mathcal{F} \) is the family of all compact subgroups of \( G \) and simply denote the ideal by \( \mathcal{VC} \) in this case.

This ideal arises if we try to compute \( G \)-equivariant homology theories in terms of \( H \)-equivariant homology theories for \( H \in \mathcal{F} \). The ideal \( \mathcal{VC} \) is closely related to the approach to the Baum–Connes assembly map in [17].

The authors feel more at home with Kasparov theory than with spectra. Many readers will prefer to work in categories of spectra of, say, \( G \)-CW-complexes. We do not introduce these categories here; but it should be clear enough that they support similar restriction functors, which provide analogues of the ideals \( \mathcal{VC}_\mathcal{F} \).

**Example 7.** Let \( G \) and \( \mathcal{IC}^G \) be as in Example 6. Using the crossed product functor (also called descent functor)

\[
G \rtimes \omega : \mathcal{IC}^G \to \mathcal{IC}, \quad A \mapsto G \rtimes A,
\]

we define ideals \( \mathcal{IK}_\omega \subseteq \mathcal{IK} \subseteq \mathcal{IC} \) as in (1.2) and (1.3) by

\[
\mathcal{IK}_\omega(A, B) := \{ f \in \mathcal{IC}^G(A, B) \mid G \rtimes f = 0 \in \mathcal{IC}(G \rtimes A, G \rtimes B) \},
\]

\[
\mathcal{IK}(A, B) := \{ f \in \mathcal{IC}^G(A, B) \mid \mathcal{IK}_\omega(G \rtimes f) = 0 : \mathcal{IK}_\omega(G \rtimes A) \to \mathcal{IK}_\omega(G \rtimes B) \}.
\]

We only study these ideals for compact \( G \). In this case, the Green–Julg Theorem identifies \( \mathcal{IK}_\omega(G \rtimes A) \) with the \( G \)-equivariant K-theory \( \mathcal{IK}_G^G(A) \) (see [11]). Hence the ideal \( \mathcal{IK} \subseteq \mathcal{IC} \) is a good equivariant analogue of the ideal \( \mathcal{IK} \) in \( \mathcal{IC} \).

Literally the same definition as above provides ideals \( \mathcal{IK}_\omega \subseteq \mathcal{IK} \subseteq \mathcal{IC} \) if \( G \) is a compact quantum group. We will always allow this more general situation below, but readers unfamiliar with quantum groups may ignore this.

**Remark 8.** We emphasise quantum groups here because Examples 6 and 7 become closely related in this context. This requires a quantum group analogue of the ideals \( \mathcal{VC}_\mathcal{F} \) in \( \mathcal{IC}^G \) of Example 6. If \( G \) is a locally compact quantum group, then Saad Baaj and Georges Skandalis construct a \( G \)-equivariant Kasparov category \( \mathcal{IC}^G \) in [2]. There is a forgetful functor \( \text{Res}^H_H : \mathcal{IC}^G \to \mathcal{IC}^H \) for each closed quantum subgroup \( H \subseteq G \). Therefore, a family \( \mathcal{F} \) of closed quantum subgroups yields an ideal \( \mathcal{VC}_\mathcal{F} \) in \( \mathcal{IC}^G \) as in Example 6.

Let \( G \) be a compact group as in Example 7. Any crossed product \( G \rtimes A \) carries a canonical coaction of \( G \), that is, a coaction of the discrete quantum group \( C^*(G) \). Baaj–Skandalis duality asserts that this yields an equivalence of categories \( \mathcal{IC}^G \cong \mathcal{IC}^{C^*(G)} \) (see [2]). We get back the crossed product functor \( \mathcal{IC}^G \to \mathcal{IC} \)
by composing this equivalence with the restriction functor \(KK^C(G) \to KK\) for the trivial quantum subgroup. Hence \(\mathcal{F}_x \subseteq KK^G\) corresponds by Baaj-Skandalis duality to \(\mathcal{V}_F \subseteq KK^C(G)\), where \(F\) consists only of the trivial quantum subgroup. Thus the constructions in Examples 6 and 7 are both special cases of a more general construction for locally compact quantum groups.

Finally, we consider a classical example from homological algebra.

**Example 9.** Let \(\mathcal{C}\) be an Abelian category. Let \(\mathcal{H}(A)\) be the homotopy category of unbounded chain complexes

\[
\cdots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} C_{n-2} \xrightarrow{\delta_{n-2}} \cdots
\]

over \(\mathcal{C}\). The space of morphisms \(A \to B\) in \(\mathcal{H}(\mathcal{C})\) is the space \([A,B]\) of homotopy classes of chain maps from \(A\) to \(B\).

Taking homology defines functors \(H_n: \mathcal{H}(\mathcal{C}) \to \mathcal{C}\) for \(n \in \mathbb{Z}\), which we combine to a single functor \(H: \mathcal{H}(\mathcal{C}) \to \mathcal{C}^\mathbb{Z}\). We let \(\mathcal{I}_H \subseteq \mathcal{H}(\mathcal{C})\) be its kernel:

\[
(2.2) \quad \mathcal{I}_H(A,B) := \{ f \in [A,B] \mid H_n(f) = 0 \}.
\]

We also consider the category \(\mathcal{H}(\mathcal{C}';\mathbb{Z}/p)\) of \(p\)-periodic chain complexes over \(\mathcal{C}\) for \(p \in \mathbb{N}_{\geq 1}\); its objects satisfy \(C_n = C_{n+p}\) and \(\delta_n = \delta_{n+p}\) for all \(n \in \mathbb{Z}\), and chain maps and homotopies are required \(p\)-periodic as well. The category \(\mathcal{H}(\mathcal{C}';\mathbb{Z}/2)\) plays a role in connection with cyclic cohomology, especially with local cyclic cohomology (14,21). The category \(\mathcal{H}(\mathcal{C}';\mathbb{Z}/1)\) is isomorphic to the category of chain complexes without grading. By convention, we let \(\mathbb{Z}/0 = \mathbb{Z}\), so that \(\mathcal{H}(\mathcal{C}';\mathbb{Z}/0) = \mathcal{H}(\mathcal{C})\).

The homology of a periodic chain complex is, of course, periodic, so that we get a homological functor \(H: \mathcal{H}(\mathcal{C}';\mathbb{Z}/p) \to \mathcal{C}^\mathbb{Z}/p\); here \(\mathcal{C}^\mathbb{Z}/p\) denotes the category of \(\mathbb{Z}/p\)-graded objects of \(\mathcal{C}\). We let \(\mathcal{I}_H \subseteq \mathcal{H}(\mathcal{C}';\mathbb{Z}/p)\) be the kernel of \(H\) as in (2.2).

### 2.3. What is a triangulated category?

A triangulated category is a category \(\mathcal{T}\) with a suspension automorphism \(\Sigma: \mathcal{T} \to \mathcal{T}\) and a class of exact triangles, subject to various axioms (see 17,19,25). An exact triangle is a diagram in \(\mathcal{T}\) of the form

\[
A \to B \to C \to \Sigma A \text{ or } A \xrightarrow{[1]} B \xrightarrow{\gamma} C,
\]

where the \([1]\) in the arrow \(C \to A\) warns us that this map has degree 1. A morphism of triangles is a triple of maps \(\alpha, \beta, \gamma\) making the obvious diagram commute.

A typical example is the homotopy category \(\mathcal{H}(\mathcal{C}';\mathbb{Z}/p)\) of \(\mathbb{Z}/p\)-graded chain complexes. Here the suspension functor is the (signed) translation functor

\[
\Sigma((C_n, d_n)) := (C_{n-1}, -d_{n-1}) \quad \text{on objects},
\]

\[
\Sigma((f_n)) := (f_{n-1}) \quad \text{on morphisms};
\]

a triangle is exact if it is isomorphic to a mapping cone triangle

\[
A \xrightarrow{f} B \to \text{cone}(f) \to \Sigma A
\]

for some chain map \(f\); the maps \(B \to \text{cone}(f) \to \Sigma A\) are the canonical ones. It is well-known that this defines a triangulated category for \(p = 0\); the arguments for \(p \geq 1\) are essentially the same.

Another classical example is the stable homotopy category, say, of compactly generated pointed topological spaces (it is not particularly relevant which category
of spaces or spectra we use). The suspension is \( \Sigma(A) := S^1 \wedge A \); a triangle is exact if it is isomorphic to a \textit{mapping cone triangle}

\[
A \xrightarrow{f} B \rightarrow \text{cone}(f) \rightarrow \Sigma A
\]

for some map \( f \); the maps \( B \rightarrow \text{cone}(f) \rightarrow \Sigma A \) are the canonical ones.

We are mainly interested in the categories \( \text{KK} \) and \( \text{KK}^G \) introduced in \S 2.2. Their triangulated category structure is discussed in detail in [17]. We are facing a notational problem because the functor \( X \mapsto C_0(X) \) from pointed compact spaces to \( C^* \)-algebras is \textit{contravariant}, so that \textit{mapping cone triangles} now have the form

\[
A \xrightarrow{f} B \leftarrow \text{cone}(f) \leftarrow C_0(\mathbb{R}, A)
\]

for a *-homomorphism \( f : B \rightarrow A \); here

\[
\text{cone}(f) = \{(a, b) \in C_0((0, \infty], A) \times B \mid a(\infty) = f(b)\}
\]

and the maps \( C_0(\mathbb{R}, A) \rightarrow \text{cone}(f) \rightarrow B \) are the obvious ones, \( a \mapsto (a, 0) \) and \( (a, b) \mapsto b \).

It is reasonable to view a *-homomorphism from \( A \) to \( B \) as a morphism from \( B \) to \( A \). Nevertheless, we prefer the convention that an algebra homomorphism \( A \rightarrow B \) is a morphism \( A \rightarrow B \). But then the most natural triangulated category structure lives on the opposite category \( \text{KK}^{\text{op}} \). This creates only notational difficulties because the opposite category of a triangulated category inherits a canonical triangulated category structure, which has “the same” exact triangles. However, the passage to opposite categories exchanges suspensions and desuspensions and modifies some sign conventions. Thus the functor \( A \mapsto C_0(\mathbb{R}, A) \), which is the suspension functor in \( \text{KK}^{\text{op}} \), becomes the desuspension functor in \( \text{KK} \). Fortunately, Bott periodicity implies that \( \Sigma^2 \cong \text{id} \), so that \( \Sigma \) and \( \Sigma^{-1} \) agree.

Depending on your definition of a triangulated category, you may want the suspension to be an equivalence or isomorphism of categories. In the latter case, you must replace \( \text{KK}^G \) by an equivalent category (see [17]); since this is not important here, we do not bother about this issue.

A triangle in \( \text{KK}^G \) is called \textit{exact} if it is isomorphic to a mapping cone triangle

\[
C_0(\mathbb{R}, B) \rightarrow \text{cone}(f) \rightarrow A \xrightarrow{f} B
\]

for some (equivariant) *-homomorphism \( f \).

An important source of exact triangles in \( \text{KK}^G \) are \textit{extensions}. If \( A \rightarrow B \rightarrow C \) is an extension of \( G \)-\( C^* \)-algebras with an equivariant completely positive contractive section, then it yields a class in \( \text{Ext}(C, A) \cong \text{KK}(\Sigma^{-1}C, A) \); the resulting triangle

\[
\Sigma^{-1}C \rightarrow A \rightarrow B \rightarrow C
\]

in \( \text{KK}^G \) is exact and called an \textit{extension triangle}. It is easy to see that any exact triangle is isomorphic to an extension triangle.

It is shown in [17] that \( \text{KK} \) and \( \text{KK}^G \) for a locally compact group \( G \) are triangulated categories with this extra structure. The same holds for the equivariant Kasparov theory \( \text{KK}^S \) with respect to any \( C^* \)-bialgebra \( S \); this theory was defined by Baaj and Skandalis in [2].

The triangulated category axioms are discussed in greater detail in [17, 19, 25]. They encode some standard machinery for manipulating long exact sequences. Most of them amount to formal properties of mapping cones and mapping cylinders,
which we can prove as in classical topology. The only axiom that requires more care
is that any morphism \( f : A \to B \) should be part of an exact triangle.

Unlike in [17], we prefer to construct this triangle as an extension triangle because
this works in greater generality; we have taken this idea from Radu Popescu and
Alexander Bonkat ([6, 20]). Any element in \( \text{KK}^S(A, B) \) can be represented by an extension \( \mathbb{K}(\mathcal{H}) \to E \to A \) with an equivariant completely
positive contractive section, where \( \mathcal{H} \) is a full \( S \)-equivariant Hilbert \( C_0(\mathbb{R}, B) \)-module,
so that \( \mathbb{K}(\mathcal{H}) \) is \( \text{KK}^S \)-equivalent to \( C_0(\mathbb{R}, B) \). Hence the resulting extension triangle
in \( \text{KK}^S \) is isomorphic to one of the form

\[
C_0(\mathbb{R}, A) \to C_0(\mathbb{R}, B) \to E \to A;
\]

by construction, it contains the suspension of the given class in \( \text{KK}^S_0(A, B) \); it is
easy to remove the suspension.

**Definition 10.** Let \( \mathfrak{T} \) be a triangulated and \( \mathfrak{C} \) an Abelian category. A covariant
functor \( F : \mathfrak{T} \to \mathfrak{C} \) is called *homological* if \( F(A) \to F(B) \to F(C) \) is exact at \( F(B) \)
for all exact triangles \( A \to B \to C \to \Sigma A \). A contravariant functor with the analogous exactness property is called *cohomological*.

Let \( A \to B \to C \to \Sigma A \) be an exact triangle. Then a homological functor
\( F : \mathfrak{T} \to \mathfrak{C} \) yields a natural long exact sequence

\[
\cdots \to F_{n+1}(C) \to F_n(A) \to F_n(B) \to F_{n-1}(C) \to F_{n-1}(A) \to F_{n-1}(B) \to \cdots
\]

with \( F_n(A) := F(\Sigma^{-n} A) \) for \( n \in \mathbb{Z} \), and a cohomological functor \( F : \mathfrak{T}^{\text{op}} \to \mathfrak{C} \) yields a
natural long exact sequence

\[
\cdots \leftarrow F^{n+1}(C) \leftarrow F^n(A) \leftarrow F^n(B) \leftarrow F^{n-1}(C) \leftarrow F^{n-1}(A) \leftarrow F^{n-1}(B) \leftarrow \cdots
\]

with \( F^n(A) := F(\Sigma^{-n} A) \).

**Proposition 11.** Let \( \mathfrak{T} \) be a triangulated category. The functors
\( \mathfrak{T}(\_ , A) : \mathfrak{T} \to \mathfrak{Ab}, \quad B \mapsto \mathfrak{T}(A, B) \)
are homological for all \( A \in \mathfrak{T} \). Dually, the functors
\( \mathfrak{T}(A, \_ ) : \mathfrak{T}^{\text{op}} \to \mathfrak{Ab}, \quad A \mapsto \mathfrak{T}(A, B) \)
are cohomological for all \( B \in \mathfrak{T} \).

Observe that

\[
\mathfrak{T}^n(A, B) = \mathfrak{T}(\Sigma^{-n} A, B) \cong \mathfrak{T}(A, \Sigma^n B) \cong \mathfrak{T}_{-n}(A, B).
\]

**Definition 12.** A *stable additive category* is an additive category equipped with
an (additive) automorphism \( \Sigma \), called *suspension*.

A *stable homological functor* is a homological functor \( F : \mathfrak{T} \to \mathfrak{C} \) into a stable
Abelian category \( \mathfrak{C} \) together with natural isomorphisms \( F(\Sigma\mathfrak{T}(A)) \cong \Sigma\mathfrak{C}(F(A)) \) for
all \( A \in \mathfrak{T} \).

**Example 13.** The category \( \mathfrak{C}^{\mathbb{Z}/p} \) of \( \mathbb{Z}/p \)-graded objects of an Abelian category \( \mathfrak{C} \) is
stable for any \( p \in \mathbb{N} \); the suspension automorphism merely shifts the grading. The
functors \( K_\_ : \text{KK} \to \mathfrak{Ab}^{\mathbb{Z}/2} \) and \( H_\_ : \mathfrak{Ab}(\mathfrak{C}; \mathbb{Z}/p) \to \mathfrak{C}^{\mathbb{Z}/p} \) introduced in Examples [5] and [3] are stable homological functors.
If $F: \mathcal{T} \to \mathcal{C}$ is any homological functor, then

$$F_n : \mathcal{T} \to \mathcal{C}^Z, \quad A \mapsto (F_n(A))_{n \in \mathbb{Z}}$$

is a stable homological functor. Many of our examples satisfy Bott periodicity, that is, there is a natural isomorphism $F_2(A) \cong F(A)$. Then we get a stable homological functor $F_2 : \mathcal{T} \to \mathcal{C}^{Z/2}$. A typical example for this is the functor $K_k$.

**Definition 14.** A functor $F: \mathcal{T} \to \mathcal{T}'$ between two triangulated categories is called exact if it intertwines the suspension automorphisms (up to specified natural isomorphisms) and maps exact triangles in $\mathcal{T}$ again to exact triangles in $\mathcal{T}'$.

**Example 15.** The restriction functor $\text{Res}_G^H : \text{KK}^G \to \text{KK}^H$ for a closed quantum subgroup $H$ of a locally compact quantum group $G$ and the crossed product functors $G \ltimes \omega, G \rtimes \omega: \text{KK}^G \to \text{KK}$ are exact because they preserve mapping cone triangles.

Let $F: \mathcal{T}_1 \to \mathcal{T}_2$ be an exact functor. If $G: \mathcal{T}_2 \to?\mathcal{T}_3$ is exact, homological, or cohomological, then so is $G \circ F$.

Using Examples 13 and 15 we see that the functors that define the ideals $\ker \gamma$ in Example 7, $V_C$, and $J_{\mathbb{K}, \mathbb{K}}$ in Example 7, and $J_H$ in Example 9 are all stable and either homological or exact.

2.4. **The universal homological functor.** The following general construction of Peter Freyd ([10]) plays an important role in [3]. For an additive category $\mathcal{C}$, let $\mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$ be the category of contravariant additive functors $\mathcal{C} \to \mathbb{Ab}$, with natural transformations as morphisms. Unless $\mathcal{C}$ is essentially small, this is not quite a category because the morphisms may form classes instead of sets. We may ignore this set-theoretic problem because the bivariant Kasparov categories that we are interested in are essentially small, and the subcategory $\text{Coh}(\mathcal{C})$ of $\mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$ that we are going to use later on is an honest category for any $\mathcal{C}$.

The category $\mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$ is Abelian: if $f: F_1 \to F_2$ is a natural transformation, then its kernel, cokernel, image, and co-image are computed pointwise on the objects of $\mathcal{C}$, so that they boil down to the corresponding constructions with Abelian groups.

The Yoneda embedding is an additive functor

$$Y: \mathcal{C} \to \mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab}), \quad B \mapsto \mathcal{T}(\omega, B).$$

This functor is fully faithful, and there are natural isomorphisms

$$\text{Hom}(Y(B), F) \cong F(B) \quad \text{for all } F \in \mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab}), \ B \in \mathcal{T}$$

by the Yoneda Lemma. A functor $F \in \mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$ is called representable if it is isomorphic to $Y(B)$ for some $B \in \mathcal{C}$. Hence $Y$ yields an equivalence of categories between $\mathcal{C}$ and the subcategory of representable functors in $\mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$.

A functor $F \in \mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$ is called finitely presented if there is an exact sequence $Y(B_1) \to Y(B_2) \to F \to 0$ with $B_1, B_2 \in \mathcal{T}$. Since $Y$ is fully faithful, this means that $F$ is the cokernel of $Y(f)$ for a morphism $f$ in $\mathcal{C}$. We let $\text{Coh}(\mathcal{C})$ be the full subcategory of finitely presented functors in $\mathfrak{Fun}(\mathcal{C}^{\text{op}}, \mathbb{Ab})$. Since representable functors belong to $\text{Coh}(\mathcal{C})$, we still have a Yoneda embedding $Y: \mathcal{C} \to \text{Coh}(\mathcal{C})$.

Although the category $\text{Coh}(\mathcal{T})$ tends to be very big and therefore unwieldy, it plays an important theoretical role.

**Theorem 16** (Freyd’s Theorem). Let $\mathcal{T}$ be a triangulated category.

Then $\text{Coh}(\mathcal{T})$ is a stable Abelian category that has enough projective and enough injective objects, and the projective and injective objects coincide.
The functor \( \mathcal{Y} : \mathcal{T} \to \text{Coh}(\mathcal{T}) \) is fully faithful, stable, and homological. Its essential range \( \mathcal{Y}(\mathcal{T}) \) consists of projective-injective objects. Conversely, an object of \( \text{Coh}(\mathcal{T}) \) is projective-injective if and only if it is a retract of an object of \( \mathcal{Y}(\mathcal{T}) \).

The functor \( \mathcal{Y} \) is the universal (stable) homological functor in the following sense: any (stable) homological functor \( F : \mathcal{T} \to \mathcal{C} \) to a (stable) Abelian category \( \mathcal{C} \) factors uniquely as \( F = \tilde{F} \circ \mathcal{Y} \) for a (stable) exact functor \( \tilde{F} : \text{Coh}(\mathcal{T}) \to \mathcal{C} \).

If idempotents in \( \mathcal{T} \) split – as in all our examples – then \( \mathcal{Y}(\mathcal{T}) \) is closed under retracts, so that \( \mathcal{Y}(\mathcal{T}) \) is equal to the class of projective-injective objects in \( \text{Coh}(\mathcal{T}) \).

2.5. Homological ideals in triangulated categories. Let \( \mathcal{T} \) be a triangulated category, let \( \mathcal{C} \) be a stable additive category, and let \( F : \mathcal{T} \to \mathcal{C} \) be a stable homological functor. Then \( \ker F \) is a stable ideal in the following sense:

**Definition 17.** An ideal \( \mathcal{I} \) in \( \mathcal{T} \) is called *stable* if the suspension isomorphisms \( \Sigma : \mathcal{T}(A, B) \xrightarrow{\cong} \mathcal{T}(\Sigma A, \Sigma B) \) for \( A, B \in \mathcal{T} \) restrict to isomorphisms
\[
\Sigma : \mathcal{I}(A, B) \xrightarrow{\cong} \mathcal{I}(\Sigma A, \Sigma B).
\]

If \( \mathcal{I} \) is stable, then there is a unique suspension automorphism on \( \mathcal{T}/\mathcal{I} \) for which the canonical functor \( \mathcal{T} \to \mathcal{T}/\mathcal{I} \) is stable. Thus the stable ideals are exactly the kernels of stable additive functors.

**Definition 18.** An ideal \( \mathcal{I} \) in a triangulated category \( \mathcal{T} \) is called *homological* if it is the kernel of a stable homological functor.

**Remark 19.** Freyd’s Theorem shows that \( \mathcal{Y} \) induces a bijection between (stable) exact functors \( \text{Coh}(\mathcal{T}) \to \mathcal{C} \) and (stable) homological functors \( \mathcal{T} \to \mathcal{C} \) because \( \tilde{F} \circ \mathcal{Y} \) is homological if \( \tilde{F} : \text{Coh}(\mathcal{T}) \to \mathcal{C} \) is exact. Hence the notion of homological functor is independent of the triangulated category structure on \( \mathcal{T} \) because the Yoneda embedding \( \mathcal{Y} : \mathcal{T} \to \text{Coh}(\mathcal{T}) \) does not involve any additional structure. Hence the notion of homological ideal only uses the suspension automorphism, not the class of exact triangles.

All the ideals considered in [2.2] except for \( \ker \kappa \) in Example 5 are kernels of stable homological functors or exact functors. Those of the first kind are homological by definition. If \( F : \mathcal{T} \to \mathcal{T}’ \) is an exact functor between two triangulated categories, then \( \mathcal{Y} \circ F : \mathcal{T} \to \text{Coh}(\mathcal{T}) \) is a stable homological functor with \( \ker \mathcal{Y} \circ F = \ker F \) by Freyd’s Theorem [16]. Hence kernels of exact functors are homological as well.

Is any homological ideal the kernel of an exact functor? This is not the case:

**Proposition 20.** Let \( \text{Der}(\text{Ab}) \) be the derived category of the category \( \text{Ab} \) of Abelian groups. Define the ideal \( \mathcal{I}_H \) in \( \text{Der}(\text{Ab}) \) as in Example 7. This ideal is not the kernel of an exact functor.

We postpone the proof to the end of [3.1] because it uses the machinery of [3.1].

It takes some effort to characterise homological ideals because \( \mathcal{T}/\mathcal{I} \) is almost never Abelian. The results in [1], §2–3 show that an ideal is homological if and only if it is *saturated* in the notation of [4]. We do not discuss this notion here because most ideals that we consider are obviously homological. The only example where we could profit from an abstract characterisation is the ideal \( \ker \kappa \) in Example 5.

There is no obvious homological functor whose kernel is \( \ker \kappa \) because \( \ker \kappa \) is not a functor on KK. Nevertheless, \( \ker \kappa \) is the kernel of an exact functor; the relevant functor is the functor \( \text{Ker} \to \text{UCT} \), where UCT is the variant of KK that satisfies
the Universal Coefficient Theorem in complete generality. This functor can be constructed as a localisation of KK (see [17]). The Universal Coefficient Theorem implies that its kernel is exactly $\ker \kappa$.

3. From homological ideals to derived functors

Once we have a stable homological functor $F : \mathcal{T} \to \mathcal{C}$, it is not surprising that we can do a certain amount of homological algebra in $\mathcal{T}$. For instance, we may call a chain complex of objects of $\mathcal{T}$ $F$-exact if $F$ maps it to an exact chain complex in $\mathcal{C}$; and we may call an object $F$-projective if $F$ maps it to a projective object in $\mathcal{C}$. But are these definitions reasonable?

We propose that a reasonable homological notion should depend only on the ideal $\ker F$. We will see that the notion of $F$-exact chain complex is reasonable and only depends on $\ker F$. In contrast, the notion of projectivity above depends on $F$ and is only reasonable in special cases. There is another, more useful, notion of projective object that depends only on the ideal $\ker F$.

Various notions from homological algebra still make sense in the context of homological ideals in triangulated categories. Our discussion mostly follows [1, 4, 8, 9]. All our definitions involve only the ideal, not a stable homological functor that defines it. We reformulate them in terms of an exact or a stable homological functor defining the ideal in order to understand what they mean in concrete cases. Following [2], we construct projective objects using adjoint functors.

The most sophisticated concept in this section is the universal $\mathcal{I}$-exact functor, which gives us complete control over projective resolutions and derived functors. We can usually describe such functors very concretely.

3.1. Basic notions. We introduce some useful terminology related to an ideal:

**Definition 21.** Let $\mathcal{I}$ be a homological ideal in a triangulated category $\mathcal{T}$.

- Let $f : A \to B$ be a morphism in $\mathcal{T}$; embed it in an exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$. We call $f$
  - $\mathcal{I}$-monic if $h \in \mathcal{I}$;
  - $\mathcal{I}$-epic if $g \in \mathcal{I}$;
  - an $\mathcal{I}$-equivalence if it is both $\mathcal{I}$-monic and $\mathcal{I}$-epic, that is, $g, h \in \mathcal{I}$;
  - an $\mathcal{I}$-phantom map if $f \in \mathcal{I}$.

- An object $A \in \mathcal{T}$ is called $\mathcal{I}$-contractible if $\text{id}_A \in \mathcal{I}(A, A)$.

- An exact triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathcal{T}$ is called $\mathcal{I}$-exact if $h \in \mathcal{I}$.

The notions of monomorphism (or monic morphism) and epimorphism (or epic morphism) – which can be found in any book on category theory such as [13] – are categorical ways to express injectivity or surjectivity of maps. A morphism in an Abelian category that is both monic and epic is invertible.

The classes of $\mathcal{I}$-phantom maps, $\mathcal{I}$-monics, $\mathcal{I}$-epics, and of $\mathcal{I}$-exact triangles determine each other uniquely because we can embed any morphism in an exact triangle in any position. It is a matter of taste which of these is considered most fundamental. Following Daniel Christensen ([8]), we favour the phantom maps. Other authors prefer exact triangles instead ([1, 4, 9]). Of course, the notion of an $\mathcal{I}$-phantom map is redundant; it becomes more relevant if we consider, say, the class of $\mathcal{I}$-exact triangles as our basic notion.
Notice that \( f \) is \( \mathcal{I} \)-epic or \( \mathcal{I} \)-monic if and only if \(-f\) is. If \( f \) is \( \mathcal{I} \)-epic or \( \mathcal{I} \)-monic, then so are \( \Sigma^n(f) \) for all \( n \in \mathbb{Z} \) because \( \mathcal{I} \) is stable. Similarly, (signed) suspensions of \( \mathcal{I} \)-exact triangles remain \( \mathcal{I} \)-exact triangles.

**Lemma 22.** Let \( F: \mathcal{I} \to \mathcal{C} \) be a stable homological functor into a stable Abelian category \( \mathcal{C} \).

- A morphism \( f \) in \( \mathcal{I} \) is
  - a ker \( F \)-phantom map if and only if \( F(f) = 0 \);
  - ker \( F \)-monic if and only if \( F(f) \) is monic;
  - ker \( F \)-epic if and only if \( F(f) \) is epic;
  - a ker \( F \)-equivalence if and only if \( F(f) \) is invertible.
- An object \( A \in \mathcal{I} \) is ker \( F \)-contractible if and only if \( F(A) = 0 \).
- An exact triangle \( A \to B \to C \to \Sigma A \) is ker \( F \)-exact if and only if
  \[
  0 \to F(A) \to F(B) \to F(C) \to 0
  \]
  is a short exact sequence in \( \mathcal{C} \).

**Proof.** Sequences in \( \mathcal{C} \) of the form \( X \xrightarrow{0} Y \xrightarrow{f} Z \) or \( X \xrightarrow{f} Y \xrightarrow{0} Z \) are exact at \( Y \) if and only if \( f \) is monic or epic, respectively. Moreover, a sequence of the form \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \) is exact if and only if \( 0 \to Y \to Z \to U \to 0 \) is exact.

Combined with the long exact homology sequences for \( F \) and suitable exact triangles, these observations yield the assertions about monomorphisms, epimorphisms, and exact triangles. The description of equivalences and contractible objects follows, and phantom maps are trivial, anyway.

Now we specialise these notions to the ideal \( \mathcal{I}_K \subseteq KK \) of Example 5, replacing \( \mathcal{I}_K \) by \( K \) in our notation to avoid clutter.

- Let \( f \in KK(A,B) \) and let \( K_*(f): K_*(A) \to K_*(B) \) be the induced map. Then \( f \) is
  - a \( K \)-phantom map if and only if \( K_*(f) = 0 \);
  - \( K \)-monic if and only if \( K_*(f) \) is injective;
  - \( K \)-epic if and only if \( K_*(f) \) is surjective;
  - a \( K \)-equivalence if and only if \( K_*(f) \) is invertible.
- A \( C^* \)-algebra \( A \in KK \) is \( K \)-contractible if and only if \( K_*(A) = 0 \).
- An exact triangle \( A \to B \to C \to \Sigma A \) in \( KK \) is \( K \)-exact if and only if
  \[
  0 \to K_*(A) \to K_*(B) \to K_*(C) \to 0
  \]
  is a short exact sequence (of \( \mathbb{Z}/2 \)-graded Abelian groups).

Similar things happen for the other ideals in [2.2] that are naturally defined as kernels of stable homological functors.

**Remark 23.** It is crucial for the above theory that we consider functors that are both stable and homological. Everything fails if we drop either assumption and consider functors such as \( K_0(A) \) or \( \text{Hom}(\mathbb{Z}/4, K_*(A)) \).

**Lemma 24.** An object \( A \in \mathcal{I} \) is \( \mathcal{I} \)-contractible if and only if \( 0 \to A \) is an \( \mathcal{I} \)-equivalence. A morphism \( f \) in \( \mathcal{I} \) is an \( \mathcal{I} \)-equivalence if and only if its generalised mapping cone is \( \mathcal{I} \)-contractible.

Thus the classes of \( \mathcal{I} \)-equivalences and of \( \mathcal{I} \)-contractible objects determine each other. But they do not allow us to recover the ideal itself. For instance, the ideals \( \mathcal{I}_K \) and \( \ker \kappa \) in Example 4 have the same contractible objects and equivalences.
Proof. Recall that the generalised mapping cone of \( f \) is the object \( C \) that fits in an exact triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \). The long exact sequence for this triangle yields that \( F(f) \) is invertible if and only if \( F(C) = 0 \), where \( F \) is some stable homological functor \( F \) with \( \ker F = \mathcal{I} \). Now the second assertion follows from Lemma 22. Since the generalised mapping cone of \( 0 \to A \) is \( A \), the first assertion is a special case of the second one.

Many ideals are defined as \( \ker F \) for an exact functor \( F : \mathcal{I} \to \mathcal{I}' \) between triangulated categories. We can also use such a functor to describe the above notions:

Lemma 25. Let \( \mathcal{I} \) and \( \mathcal{I}' \) be triangulated categories and let \( F : \mathcal{I} \to \mathcal{I}' \) be an exact functor.

- An object \( A \in \mathcal{I} \) is \( F \)-contractible if and only if \( F(A) = 0 \).
- An exact triangle \( A \to B \to C \to \Sigma A \) is \( F \)-exact if and only if the exact triangle \( F(A) \to F(B) \to F(C) \to F(\Sigma A) \) in \( \mathcal{I}' \) splits.

We will explain the notation during the proof.

Proof. A morphism \( f : X \to Y \) in \( \mathcal{I}' \) is called split epic (split monic) if there is \( g : Y \to X \) with \( f \circ g = \text{id}_Y \) (\( g \circ f = \text{id}_X \)). An exact triangle \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \) is said to split if \( h = 0 \). This immediately yields the characterisation of \( \ker F \)-exact triangles. Any split triangle is isomorphic to a direct sum triangle, so that \( f \) is split monic and \( g \) is split epic (\cite{KK} Corollary 1.2.7). Conversely, either of these conditions implies that the triangle is split.

Since the \( \ker F \)-exact triangles determine the \( \ker F \)-epimorphisms and \( \ker F \)-monomorphisms, the latter are detected by \( F(f) \) being split epic or split monic, respectively. It is clear that split epimorphisms and split monomorphisms are epimorphisms and monomorphisms, respectively. The converse holds in a triangulated category because if we embed a monomorphism or epimorphism in an exact triangle, then one of the maps is forced to vanish, so that the exact triangle splits.

Finally, a morphism is invertible if and only if it is both split monic and split epic, and the zero map \( F(A) \to F(A) \) is invertible if and only if \( F(A) = 0 \).

Alternatively, we may prove Lemma 22 using the Yoneda embedding \( \mathcal{Y} : \mathcal{I}' \to \mathbf{Coh}(\mathcal{I}') \). The assertions about phantom maps, equivalences, and contractibility boil down to the observation that \( \mathcal{Y} \) is fully faithful. The assertions about monomorphisms and epimorphisms follow because a map \( f : A \to B \) in \( \mathcal{I}' \) becomes epic (monic) in \( \mathbf{Coh}(\mathcal{I}') \) if and only if it is split epic (monic) in \( \mathcal{I}' \).

There is a similar description for \( \bigcap \ker F_i \) for a set \( \{ F_i \} \) of exact functors. This applies to the ideal \( \mathcal{V} \mathcal{C}_F \) for a family of (quantum) subgroups \( F \) in a locally compact (quantum) group \( G \) (Example 6). Replacing \( \mathcal{V} \mathcal{C}_F \) by \( F \) in our notation to avoid clutter, we get:

- A morphism \( f \in \mathbf{KK}^G(A, B) \) is
  - an \( F \)-phantom map if and only if \( \text{Res}_H^H(f) = 0 \) in \( \mathbf{KK}^H \) for all \( H \in F \);
The generator definition differs from Beligiannis’ one (1,4), which we recall first.

Deﬁned for exact triangles so far, will now be extended to chain complexes. Our

 Exact chain complexes.

3.2.

Gennadi Kasparov (see [17]). These examples show that this question is subtle and

the proof of the Baum–Connes Conjecture for these groups by Nigel Higson and

an amenable group, then

\( VC \)

fundamental group, then

\( G \)

compact quantum group. In contrast, if

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because the Universal Coeﬃcient Theorem does not hold for arbitrary separable

questions are equivalent by Lemma 24. The answer is negative for the ideal

\( I \)

objects vanish or, equivalently, whether all

\( \text{Res}_H^G(f) \)

is the natural projection. We get an exact triangle

\( Z/2 \to Z/4 \to Z/2 \to Z/2[1] \)

in \( \text{Der}(\mathfrak{A} \mathfrak{b}) \). This triangle is \( \mathcal{J}_H \)-exact because the map \( Z/2 \to Z/4 \) is injective as a group homomorphism and hence \( \mathcal{J}_H \)-monic in \( \text{Der}(\mathfrak{A} \mathfrak{b}) \).

Assume there were an exact functor \( F: \text{Der}(\mathfrak{A} \mathfrak{b}) \to \mathfrak{T} \) with \( \ker F = \mathcal{J}_H \). Then

\( F(\tau) = 0 \), so that \( F \) maps our triangle to a split triangle and \( F(Z/4) \cong F(Z/2) \oplus F(Z/2) \) by Lemma 25. It follows that

\( F(2 \cdot \text{id}_{Z/4}) = 2 \cdot \text{id}_{F(Z/4)} = 0 \) because

\( 2 \cdot \text{id}_{F(Z/2)} = F(2 \cdot \text{id}_{Z/2}) = 0 \).

Hence

\( 2 \cdot \text{id}_{Z/4} \in \ker F = \mathcal{J}_H \), which is false. This contradiction shows that there is no exact functor \( F \) with \( \ker F = \mathcal{J}_H \).

One of the most interesting questions about an ideal is whether all \( \mathcal{J} \)-contractible objects vanish or, equivalently, whether all \( \mathcal{J} \)-equivaleces are invertible. These two questions are equivalent by Lemma 24. The answer is negative for the ideal \( \mathcal{J}_K \) in KK because the Universal Coeﬃcient Theorem does not hold for arbitrary separable

\( C^* \)-algebras. Therefore, we also get counterexamples for the ideal \( \mathcal{J}_{K,K} \) in KK\( ^{\mathbb{Z}} \) for a compact quantum group. In contrast, if \( G \) is a connected Lie group with torsion-free fundamental group, then \( \mathcal{J}_K \)-equivaleces in KK\( ^{\mathbb{Z}} \) are invertible (see [18]). If \( G \) is an amenable group, then \( \mathcal{V}C \)-equivalences in KK\( ^{\mathbb{Z}} \) are invertible; this follows from the proof of the Baum–Connes Conjecture for these groups by Nigel Higson and Gennadi Kasparov (see [17]). These examples show that this question is subtle and may involve diﬃcult analysis.

3.2. Exact chain complexes. The notion of \( \mathcal{J} \)-exactness, which we have only deﬁned for exact triangles so far, will now be extended to chain complexes. Our deﬁnition diﬀers from Beligiannis’ one (1,4), which we recall ﬁrst.

Let \( \mathfrak{T} \) be a triangulated category and let \( \mathcal{J} \) be a homological ideal in \( \mathfrak{T} \).

Deﬁnition 26. A chain complex

\[ C_\bullet := ( \cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \overset{d_{n-1}}{\to} C_{n-2} \to \cdots ) \]

in \( \mathfrak{T} \) is called \( \mathcal{J} \)-decomposable if there is a sequence of \( \mathcal{J} \)-exact triangles

\[ K_{n+1} \overset{g_n}{\to} C_n \overset{f_n}{\to} K_n \overset{h_n}{\to} \Sigma K_{n+1} \]
with \( d_n = g_{n-1} \circ f_n : C_n \to C_{n-1} \).

Such complexes are called \( \mathcal{I} \)-exact in \([14]\). This definition is inspired by the following well-known fact: a chain complex over an Abelian category is exact if and only if it splits into short exact sequences of the form \( K_n \to C_n \to K_{n-1} \) as in Definition \( 26 \).

We prefer another definition of exactness because we have not found a general explicit criterion for a chain complex to be \( \mathcal{I} \)-decomposable.

**Definition 27.** Let \( C_\bullet = (C_n, d_n) \) be a chain complex over \( \mathcal{I} \). For each \( n \in \mathbb{N} \), embed \( d_n \) in an exact triangle

\[
C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{f_n} X_n \xrightarrow{g_n} \Sigma C_n.
\]

We call \( C_\bullet \) \( \mathcal{I} \)-exact in degree \( n \) if the map \( X_n \xrightarrow{g_n} \Sigma C_n \xrightarrow{\Sigma f_{n+1}} \Sigma X_{n+1} \) belongs to \( \mathcal{I}(X_n, \Sigma X_{n+1}) \). This does not depend on auxiliary choices because the exact triangles in \( (3.1) \) are unique up to (non-canonical) isomorphism.

We call \( C_\bullet \) \( \mathcal{I} \)-exact if it is \( \mathcal{I} \)-exact in degree \( n \) for all \( n \in \mathbb{Z} \).

This definition is designed to make the following lemma true:

**Lemma 28.** Let \( F : \mathcal{I} \to \mathcal{C} \) be a stable homological functor into a stable Abelian category \( \mathcal{C} \) with \( \ker F = \mathcal{I} \). A chain complex \( C_\bullet \) over \( \mathcal{I} \) is \( \mathcal{I} \)-exact in degree \( n \) if and only if

\[
F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1})
\]

is exact at \( F(C_n) \).

**Proof.** The complex \( C_\bullet \) is \( \mathcal{I} \)-exact in degree \( n \) if and only if the map

\[
\Sigma^{-1} F(X_n) \xrightarrow{\Sigma^{-1} F(g_n)} F(C_n) \xrightarrow{F(f_{n+1})} F(X_{n+1})
\]

vanishes. Equivalently, the range of \( \Sigma^{-1} F(g_n) \) is contained in the kernel of \( F(f_{n+1}) \). The long exact sequences

\[
\cdots \to \Sigma^{-1} F(X_n) \xrightarrow{\Sigma^{-1} F(g_n)} F(C_n) \xrightarrow{F(d_n)} F(C_{n-1}) \to \cdots,
\]

\[
\cdots \to F(C_{n+1}) \xrightarrow{F(d_{n+1})} F(C_n) \xrightarrow{F(f_{n+1})} F(X_{n+1}) \to \cdots
\]

show that the range of \( \Sigma^{-1} F(g_n) \) and the kernel of \( F(f_{n+1}) \) are equal to the kernel of \( F(d_n) \) and the range of \( F(d_{n+1}) \), respectively. Hence \( C_\bullet \) is \( \mathcal{I} \)-exact in degree \( n \) if and only if \( \ker F(d_n) \subseteq \text{range } F(d_{n+1}) \). Since \( d_n \circ d_{n+1} = 0 \), this is equivalent to \( \ker F(d_n) = \text{range } F(d_{n+1}) \). \( \square \)

**Corollary 29.** \( \mathcal{I} \)-decomposable chain complexes are \( \mathcal{I} \)-exact.

**Proof.** Let \( F : \mathcal{I} \to \mathcal{C} \) be a stable homological functor with \( \ker F = \mathcal{I} \). If \( C_\bullet \) is \( \mathcal{I} \)-decomposable, then \( F(C_\bullet) \) is obtained by splicing short exact sequences in \( \mathcal{C} \). This implies that \( F(C_\bullet) \) is exact, so that \( C_\bullet \) is \( \mathcal{I} \)-exact by Lemma \( 28 \). \( \square \)

The converse implication in Corollary \( 29 \) fails in general (see Example \( 37 \)).

**Example 30.** For the ideal \( \mathcal{I}_K \) in \( KK \), Lemma \( 28 \) yields that a chain complex \( C_\bullet \) over \( KK \) is \( K \)-exact (in degree \( n \)) if and only if the chain complex

\[
\cdots \to K_*(C_{n+1}) \to K_*(C_n) \to K_*(C_{n-1}) \to \cdots
\]
of \( \mathbb{Z}/2 \)-graded Abelian groups is exact (in degree \( n \)). Similar remarks apply to the other ideals in \( \mathfrak{I} \) that are defined as kernels of stable homological functors.

As a trivial example, we consider the largest possible ideal \( \mathfrak{I} = \mathfrak{T} \). This ideal is defined by the zero functor. Lemma 28 or the definition yield that all chain complexes are \( \mathfrak{T} \)-exact. In contrast, it seems hard to characterise the \( \mathfrak{I} \)-decomposable chain complexes, already for \( \mathfrak{I} = \mathfrak{T} \).

Lemma 31. A chain complex of length 3

\[
\cdots \to 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \to \cdots
\]

is \( \mathfrak{I} \)-exact if and only if there are an \( \mathfrak{I} \)-exact exact triangle \( A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to \Sigma A' \) and a commuting diagram

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \xrightarrow{g'} & C' \\
\sim & & \sim & \sim & \sim \\
A & \xrightarrow{f} & B \xrightarrow{g} & C
\end{array}
\]

where the vertical maps \( \alpha, \beta, \gamma \) are \( \mathfrak{I} \)-equivalences. Furthermore, we can achieve that \( \alpha \) and \( \beta \) are identity maps.

Proof. Let \( F \) be a stable homological functor with \( \mathfrak{I} = \ker F \).

Suppose first that we are in the situation of (3.2). Lemma 22 yields that \( F(\alpha), F(\beta), \) and \( F(\gamma) \) are invertible and that \( 0 \to F(A') \to F(B') \to F(C') \to 0 \) is a short exact sequence. Hence so is \( 0 \to F(A) \to F(B) \to F(C) \to 0 \). Now Lemma 28 yields that our given chain complex is \( \mathfrak{I} \)-exact.

Conversely, suppose that we have an \( \mathfrak{I} \)-exact chain complex. By Lemma 28, this means that \( 0 \to F(A) \to F(B) \to F(C) \to 0 \) is a short exact sequence. Hence \( f : A \to B \) is \( \mathfrak{I} \)-monic. Embed \( f \) in an exact triangle \( A \to B \to C' \to \Sigma A \). Since \( f \) is \( \mathfrak{I} \)-monic, this triangle is \( \mathfrak{I} \)-exact. Let \( \alpha = \text{id}_A \) and \( \beta = \text{id}_B \). Since the functor \( \mathfrak{T}(\mathfrak{T}, C') \) is cohomological and \( g \circ f = 0 \), we can find a map \( \gamma : C' \to C \) making (3.2) commute. The functor \( F \) maps the rows of (3.2) to short exact sequences by Lemmas 28 and 22. Now the Five Lemma yields that \( F(\gamma) \) is invertible, so that \( \gamma \) is an \( \mathfrak{I} \)-equivalence.

Remark 32. Lemma 31 implies that \( \mathfrak{I} \)-exact chain complexes of length 3 are \( \mathfrak{I} \)-decomposable. We do not expect this for chain complexes of length 4. But we have not searched for a counterexample.

Which chain complexes over \( \mathfrak{T} \) are \( \mathfrak{I} \)-exact for \( \mathfrak{I} = \mathfrak{I} \) and hence for any homological ideal? The next definition provides the answer.

Definition 33. A chain complex \( C_* \) over a triangulated category is called homologically exact if \( F(C_*) \) is exact for any homological functor \( F: \mathfrak{T} \to \mathfrak{E} \).

Example 34. If \( A \to B \to C \to \Sigma A \) is an exact triangle, then the chain complex

\[
\cdots \to \Sigma^{-1} A \to \Sigma^{-1} B \to \Sigma^{-1} C \to A \to B \to C \to \Sigma A \to \Sigma B \to \Sigma C \to \cdots
\]

is homologically exact by the definition of a homological functor.

Lemma 35. Let \( F: \mathfrak{T} \to \mathfrak{T}' \) be an exact functor between two triangulated categories. Let \( C_* \) be a chain complex over \( \mathfrak{T} \). The following are equivalent:

(1) \( C_* \) is \( \ker F \)-exact in degree \( n \);
Remark 36. More generally, consider a set of exact functors $F_i: \mathcal{I} \to \mathcal{I}'$. As in the proof of the equivalence (1) $\iff$ (2) in Lemma [35], we see that a chain complex $C_\bullet$ is $\bigcap \ker F_i$-exact (in degree $n$) if and only if the chain complexes $F_i(C_\bullet)$ are exact (in degree $n$) for all $i$.

As a consequence, a chain complex $C_\bullet$ over $KK^G$ for a locally compact quantum group $G$ is $\mathcal{F}$-exact if and only if $\text{Res}_H^G(C_\bullet)$ is homologically exact for all $H \in \mathcal{F}$. A chain complex $C_\bullet$ over $KK^G$ for a compact quantum group $G$ is $\mathcal{J}_c$-exact if and only if the chain complex $G \ltimes C_\bullet$ over $KK$ is homologically exact.

Example 37. We exhibit an $\mathcal{I}$-exact chain complex that is not $\mathcal{J}$-decomposable for the ideal $\mathcal{I} = 0$. By Lemma [25], any 0-exact triangle is split. Therefore, a chain complex is 0-decomposable if and only if it is a direct sum of chain complexes of the form $0 \to K_n \overset{id}{\to} K_n \to 0$. Hence any decomposable chain complex is contractible and therefore mapped by any homological functor to a contractible chain complex. By the way, if idempotents in $\mathcal{I}$ split then a chain complex is 0-decomposable if and only if it is contractible.

As we have remarked in Example [34], the chain complex

$$\cdots \to \Sigma^{-1}A \to A \to B \to C \to \Sigma A \to \Sigma B \to \Sigma C \to \Sigma^2 A \to \cdots$$

is homologically exact for any exact triangle $A \to B \to C \to \Sigma A$. But such chain complexes need not be contractible. A counterexample is the exact triangle $\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to \Sigma \mathbb{Z}/2$ in $\text{Der}(\text{Ab})$, which we have already used in the proof of Proposition [20]. The resulting chain complex over $\text{Der}(\text{Ab})$ cannot be contractible because $H_\bullet$ maps it to a non-contractible chain complex.

3.2.1. More homological algebra with chain complexes. Using our notion of exactness for chain complexes, we can do homological algebra in the homotopy category $\mathcal{H}(\mathcal{I})$. We briefly sketch some results in this direction, assuming some familiarity with more advanced notions from homological algebra. We will not use this later.

The $\mathcal{I}$-exact chain complexes form a thick subcategory of $\mathcal{H}(\mathcal{I})$ because of Lemma [28]. We let $\text{Der} := \text{Der}(\mathcal{I}, \mathcal{I})$ be the localisation of $\mathcal{H}(\mathcal{I})$ at this subcategory and call it the derived category of $\mathcal{I}$ with respect to $\mathcal{I}$.

We let $\text{Der}^{\geq n}$ and $\text{Der}^{\leq n}$ be the full subcategories of $\text{Der}$ consisting of chain complexes that are $\mathcal{I}$-exact in degrees less than $n$ and greater than $n$, respectively.

Theorem 38. The pair of subcategories $\text{Der}^{\geq n}$, $\text{Der}^{\leq n}$ forms a truncation structure (t-structure) on $\text{Der}$ in the sense of [3].
This chain complex is I-exact – even homologically exact – in negative degrees, that is, \( C^{>0}_* \in \text{Der}^{>0} \). The triangulated category structure allows us to construct a chain map \( C^{>0}_* \to C_* \) that is an isomorphism on \( C_n \) for \( n \geq -1 \). Hence its mapping cone \( C^{<0}_* \) is I-exact – even contractible – in degrees \( \geq 0 \), that is, \( C^{<0}_* \in \text{Der}^{\leq -1} \).

By construction, we have an exact triangle

\[
C^{>0}_* \to C_* \to C^{<0}_* \to \Sigma C^{>0}_*
\]

in \( \text{Der} \).

We also have to check that there is no non-zero morphism \( C_* \to D_* \) in \( \text{Der} \) if \( C_* \in \text{Der}^{>0} \) and \( D_* \in \text{Der}^{\leq -1} \). Recall that morphisms in \( \text{Der} \) are represented by diagrams \( C_* \to C_* \to D_* \in H\Phi(\Sigma) \), where the first map is an I-equivalence. Hence \( C_* \in \text{Der}^{>0} \) as well. We claim that any chain map \( f: C^{>0}_* \to D^{\leq -1}_* \) is homotopic to 0. Since the maps \( C^{>0}_* \to C_* \) and \( D_* \to D^{\leq -1}_* \) are I-equivalences, any morphism \( C_* \to D_* \) vanishes in \( \text{Der} \).

It remains to prove the claim. In a first step, we use that \( D^{\leq -1}_* \) is contractible in degrees \( \geq 0 \) to replace \( f \) by a homotopic chain map supported in degrees \( < 0 \). In a second step, we use that \( C^{>0}_* \) is homologically exact in the relevant degrees to recursively construct a chain homotopy between \( f \) and 0.

Any truncation structure gives rise to an Abelian category, its core. The core of \( \text{Der} \) is the full subcategory \( \mathcal{C} \) of all chain complexes that are I-exact except in degree 0. This is a stable Abelian category, and the standard embedding \( \Sigma \to H\Phi(\Sigma) \) yields a stable homological functor \( F: \Sigma \to H\Phi(\Sigma) \) that factors uniquely as \( \Sigma \to H\Phi(\Sigma) \to \mathcal{C} \) with \( \ker F = \mathcal{I} \).

This functor is characterised uniquely by the following universal property: any (stable) homological functor \( H: \Sigma \to \mathcal{C} \) with \( \mathcal{I} \subseteq \ker H \) factors uniquely as \( H = \overline{H} \circ F \) for an exact functor \( \overline{H}: \mathcal{C} \to \mathcal{C} \).

First, we lift \( H \) to an exact functor \( H\Phi(H): H\Phi(\Sigma, \mathcal{I}) \to H\Phi(\mathcal{C}) \). Secondly, \( H\Phi(H) \) descends to a functor \( \text{Der}(H): \text{Der}(\Sigma, \mathcal{I}) \to \text{Der}(\mathcal{C}) \). Finally, \( \text{Der}(H) \) restricts to a functor \( \overline{H}: \mathcal{C} \to \mathcal{C} \) between the cores. Since \( \mathcal{I} \subseteq \ker H \), an I-exact chain complex is also ker-\( H \)-exact. Hence \( H\Phi(H) \) preserves exactness of chain complexes by Lemma [28]. This allows us to construct \( \text{Der}(H) \) and shows that \( \text{Der}(H) \) is compatible with truncation structures. This allows us to restrict it to an exact functor between the cores. Finally, we use that the core of the standard truncation structure on \( \text{Der}(\mathcal{C}) \) is \( \mathcal{C} \). It is easy to see that we have \( \overline{H} \circ F = H \).

Especially, we get an exact functor \( \text{Der}(F): \text{Der}(\Sigma, \mathcal{I}) \to \text{Der}(\mathcal{C}) \), which restricts to the identity functor \( \text{id}_\mathcal{C} \) between the cores. Hence \( \text{Der}(F) \) is fully faithful on the thick subcategory generated by \( \mathcal{C} \subseteq \text{Der}(\Sigma, \mathcal{I}) \). It seems plausible that \( \text{Der}(F) \) should be an equivalence of categories under some mild conditions on \( \mathcal{I} \).

We will continue our study of the functor \( F: \Sigma \to \mathcal{C} \) in [3.7]. The universal property determines it uniquely. Beligiannis ([4]) has another, simpler construction.

### 3.3. Projective objects.

Let \( \mathcal{I} \) be a homological ideal in a triangulated category \( \Sigma \).

**Definition 39.** A homological functor \( F: \Sigma \to \mathcal{C} \) is called I-exact if \( F(f) = 0 \) for all I-phantom maps \( f \) or, equivalently, \( \mathcal{I} \subseteq \ker F \). An object \( A \in \mathcal{C} \) is called
A \textit{projective} object is one for which the functor $T(A): T \rightarrow \mathbb{Ab}$ is exact. Dually, an object $B \in T$ is called \textit{injective} if the functor $T(\_ ; B): T \rightarrow \mathbb{Ab}^{op}$ is exact.

We write $\mathcal{P}$ for the class of $I$-projective objects in $T$.

The notions of projective and injective object are dual to each other: if we pass to the opposite category $T^{op}$ with the canonical triangulated category structure and use the same ideal $I^{op}$, then this exchanges the roles of projective and injective objects. Therefore, it suffices to discuss one of these two notions in the following.

We will only treat projective objects because all the ideals in $\mathcal{I}^2$ have enough projective objects, but most of them do not have enough injective objects.

Notice that the functor $F$ is $I$-exact if and only if the associated stable functor $F: T \rightarrow \mathbb{C}Z$ is $I$-exact because $I$ is stable. Since we require $F$ to be homological, the long exact homology sequence and Lemma 28 yield that the following conditions are all equivalent to $F$ being $I$-exact:

- $F$ maps $I$-epimorphisms to epimorphisms in $\mathbb{C}$;
- $F$ maps $I$-monomorphisms to monomorphisms in $\mathbb{C}$;
- $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$ is a short exact sequence in $\mathbb{C}$ for any $I$-exact triangle $A \rightarrow B \rightarrow C \rightarrow \Sigma A$;
- $F$ maps $I$-exact chain complexes to exact chain complexes in $\mathbb{C}$.

This specialises to equivalent definitions of $I$-projective objects.

**Lemma 40.** An object $A \in T$ is $I$-projective if and only if $I(A, B) = 0$ for all $B \in T$.

**Proof.** If $f \in I(A, B)$, then $f = f_*(\text{id}_A)$. This has to vanish if $A$ is $I$-projective. Suppose, conversely, that $I(A, B) = 0$ for all $B \in T$. If $f \in I(B, B')$, then $\Sigma(A, f)$ maps $\Sigma(A, B)$ to $\Sigma(A, B') = 0$, so that $\Sigma(A, f) = 0$. Hence $A$ is $I$-projective.

An $I$-exact functor also has the following properties (which are strictly weaker than being $I$-exact):

- $F$ maps $I$-equivalences to isomorphisms in $\mathbb{C}$;
- $F$ maps $I$-contractible objects to 0 in $\mathbb{C}$.

Again we may specialise this to $I$-projective objects.

**Lemma 41.** The class of $I$-exact homological functors $T \rightarrow \mathbb{Ab}$ or $T \rightarrow \mathbb{Ab}^{op}$ is closed under composition with $\Sigma^{\pm 1}: T \rightarrow T$, retracts, direct sums, and direct products. The class $\mathcal{P}$ of $I$-projective objects is closed under (de)suspensions, retracts, and possibly infinite direct sums (as far as they exist in $T$).

**Proof.** The first assertion follows because direct sums and products of Abelian groups are exact; the second one is a special case.

**Notation 42.** Let $\mathcal{P}$ be a set of objects of $T$. We let $(\mathcal{P})_\oplus$ be the smallest class of objects of $T$ that contains $\mathcal{P}$ and is closed under retracts and direct sums (as far as they exist in $T$).

By Lemma 41 $(\mathcal{P})_\oplus$ consists of $I$-projective objects if $\mathcal{P}$ does. We say that $\mathcal{P}$ \textit{generates all} $I$-projective objects if $(\mathcal{P})_\oplus = \mathcal{P}_I$. In examples, it is usually easier to describe a class of generators in this sense.
3.4. Projective resolutions.

**Definition 43.** Let $\mathcal{I} \subseteq \mathcal{T}$ be a homological ideal in a triangulated category and let $A \in \mathcal{T}$. A one-step $\mathcal{I}$-projective resolution is an $\mathcal{I}$-epimorphism $\pi: P \to A$ with $P \in \mathcal{P}_\mathcal{I}$. An $\mathcal{I}$-projective resolution of $A$ is an $\mathcal{I}$-exact chain complex

$$\cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A$$

with $P_n \in \mathcal{P}_\mathcal{I}$ for all $n \in \mathbb{N}$.

We say that $\mathcal{I}$ has enough projective objects if each $A \in \mathcal{T}$ has a one-step $\mathcal{I}$-projective resolution.

The following proposition contains the basic properties of projective resolutions, which are familiar from the similar situation for Abelian categories.

**Proposition 44.** If $\mathcal{I}$ has enough projective objects, then any object of $\mathcal{T}$ has an $\mathcal{I}$-projective resolution (and vice versa).

Let $P_\bullet \to A$ and $P'_\bullet \to A'$ be $\mathcal{I}$-projective resolutions. Then any map $A \to A'$ may be lifted to a chain map $P_\bullet \to P'_\bullet$, and this lifting is unique up to chain homotopy. Two $\mathcal{I}$-projective resolutions of the same object are chain homotopy equivalent. As a result, the construction of projective resolutions provides a functor

$$P: \mathcal{T} \to \mathcal{H}_\mathcal{O}(\mathcal{T}).$$

Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be an $\mathcal{I}$-exact triangle. Then there exists a canonical map $\eta: P(C) \to P(A)[1]$ in $\mathcal{H}_\mathcal{O}(\mathcal{T})$ such that the triangle

$$P(A) \xrightarrow{P(f)} P(B) \xrightarrow{P(g)} P(C) \xrightarrow{\eta} P(A)[1]$$

in $\mathcal{H}_\mathcal{O}(\mathcal{T})$ is exact; here $[1]$ denotes the translation functor in $\mathcal{H}_\mathcal{O}(\mathcal{T})$, which has nothing to do with the suspension in $\mathcal{T}$.

**Proof.** Let $A \in \mathcal{T}$. By assumption, there is a one-step $\mathcal{I}$-projective resolution $\delta_0: P_0 \to A$, which we embed in an exact triangle $A_1 \to P_0 \to A \to \Sigma A_1$. Since $\delta_0$ is $\mathcal{I}$-epic, this triangle is $\mathcal{I}$-exact. By induction, we construct a sequence of such $\mathcal{I}$-exact triangles $A_{n+1} \to P_n \to A_n \to \Sigma A_{n+1}$ for $n \in \mathbb{N}$ with $P_n \in \mathcal{P}_\mathcal{I}$ and $A_0 = A$. By composition, we obtain maps $\delta_n: P_n \to P_{n-1}$ for $n \geq 1$, which satisfy $\delta_n \circ \delta_{n+1} = 0$ for all $n \geq 0$. The resulting chain complex

$$\cdots \to P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} P_{n-2} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A \to 0$$

is $\mathcal{I}$-decomposable by construction and therefore $\mathcal{I}$-exact by Corollary 29.

The remaining assertions are proved exactly as their classical counterparts in homological algebra. We briefly sketch the arguments. Let $P_\bullet \to A$ and $P'_\bullet \to A'$ be $\mathcal{I}$-projective resolutions and let $f \in \mathcal{T}(A, A')$. We construct $f_n \in \mathcal{T}(P_n, P'_n)$ by induction on $n$ such that the diagrams

$$\begin{array}{c}
P_0 \xrightarrow{\delta_0} A \\
\downarrow f \downarrow \quad \downarrow f \\
P'_0 \xrightarrow{\delta'_0} A'
\end{array} \quad \text{and} \quad \begin{array}{c}
P_n \xrightarrow{\delta_n} P_{n-1} \\
\downarrow f_n \downarrow \quad \downarrow f_{n-1} \\
P'_n \xrightarrow{\delta'_n} P'_{n-1}
\end{array} \quad \text{for } n \geq 1$$
commute. We must check that this is possible. Since the chain complex $P'_* \to A$ is $I$-exact and $P_n$ is $I$-projective for all $n \geq 0$, the chain complexes

$$\cdots \to \mathcal{I}(P_n, P'_m) \xrightarrow{(\delta'_m)_n} \mathcal{I}(P_n, P'_{m-1}) \to \cdots \to \mathcal{I}(P_n, P'_0) \xrightarrow{(\delta'_0)_n} \mathcal{I}(P_n, A) \to 0$$

are exact for all $n \in \mathbb{N}$. This allows us to find maps $f_n$ as above. By construction, these maps form a chain map lifting $f : A \to A'$. Its uniqueness up to chain homotopy is proved similarly. If we apply this unique lifting result to two $I$-projective resolutions of the same object, we get the uniqueness of $I$-projective resolutions up to chain homotopy equivalence. Hence we get a well-defined functor $P : \mathfrak{I} \to \mathcal{H}\mathcal{O}(\mathfrak{I})$.

Now consider an $I$-exact triangle $A \to B \to C \to \Sigma A$ as in the third paragraph of the lemma. Let $X_\bullet$ be the mapping cone of some chain map $P(A) \to P(B)$ in the homotopy class $P(f)$. This chain complex is supported in degrees $\geq 0$ and has $I$-projective entries because $X_n = P(A)_{n-1} \oplus P(B)_n$. The map $X_0 = 0 \oplus P(B)_0 \to B \to C$ yields a chain map $X_\bullet \to C$, that is, the composite map $X_1 \to X_0 \to C$ vanishes. By construction, this chain map lifts the given map $B \to C$ and we have an exact triangle $P(A) \to P(B) \to X_\bullet \to P(A)[1]$ in $\mathcal{H}\mathcal{O}(\mathfrak{I})$. It remains to observe that $X_\bullet \to C$ is $I$-exact. Then $X_\bullet$ is an $I$-projective resolution of $C$. Since such resolutions are unique up to chain homotopy equivalence, we get a canonical isomorphism $X_\bullet \cong P(C)$ in $\mathcal{H}\mathcal{O}(\mathfrak{I})$ and hence the assertion in the third paragraph.

Let $F$ be a stable homological functor with $I = \ker F$. We have to check that $F(X_\bullet) \to F(C)$ is a resolution. This reduces to a well-known diagram chase in Abelian categories, using that $F(P(A)) \to F(A)$ and $F(P(B)) \to F(B)$ are resolutions and that $F(A) \to F(B) \to F(C)$ is exact.

3.5. Derived functors. We only define derived functors if there are enough projective objects because this case is rather easy and suffices for our applications. The general case can be reduced to the familiar case of Abelian categories using the results of [3.2.1]

\textbf{Definition 45.} Let $I$ be a homological ideal in a triangulated category $\mathfrak{I}$ with enough projective objects. Let $F : \mathfrak{I} \to \mathcal{C}$ be an additive functor with values in an Abelian category $\mathcal{C}$. It induces a functor $\mathcal{H}\mathcal{O}(F) : \mathcal{H}\mathcal{O}(\mathfrak{I}) \to \mathcal{H}\mathcal{O}(\mathcal{C})$, applying $F$ pointwise to chain complexes. Let $P : \mathfrak{I} \to \mathcal{H}\mathcal{O}(\mathfrak{I})$ be the projective resolution functor constructed in Proposition [44]. Let $H_n : \mathcal{H}\mathcal{O}(\mathcal{C}) \to \mathcal{C}$ be the $n$th homology functor for some $n \in \mathbb{N}$. The composite functor

$$\mathbb{L}_n F : \mathfrak{I} \xrightarrow{P} \mathcal{H}\mathcal{O}(\mathfrak{I}) \xrightarrow{\delta_0(F)} \mathcal{H}\mathcal{O}(\mathcal{C}) \xrightarrow{H_n} \mathcal{C}$$

is called the $n$th left derived functor of $F$. If $F : \mathfrak{I}^{\text{op}} \to \mathcal{C}$ is an additive functor, then the corresponding functor $H^n \circ \mathcal{H}\mathcal{O}(F) \circ P : \mathfrak{I}^{\text{op}} \to \mathcal{C}$ is denoted by $\mathbb{R}^n F$ and called the $n$th right derived functor of $F$.

More concretely, let $A \in \mathfrak{I}$ and let $(P_\bullet, \delta_\bullet)$ be an $I$-projective resolution of $A$. If $F$ is covariant, then $\mathbb{L}_n F(A)$ is the homology at $F(P_n)$ of the chain complex

$$\cdots \to F(P_{n+1}) \xrightarrow{F(\delta_{n+1})} F(P_n) \xrightarrow{F(\delta_n)} F(P_{n-1}) \to \cdots \to F(P_0) \to 0.$$

If $F$ is contravariant, then $\mathbb{R}^n F(A)$ is the cohomology at $F(P_n)$ of the cochain complex

$$\cdots \leftarrow F(P_{n+1}) \xleftarrow{F(\delta_{n+1})} F(P_n) \xleftarrow{F(\delta_n)} F(P_{n-1}) \leftarrow \cdots \leftarrow F(P_0) \leftarrow 0.$$
Lemma 46. Let $A \to B \to C \to \Sigma A$ be an $\mathcal{I}$-exact triangle. If $F : \mathcal{I} \to \mathcal{C}$ is a covariant additive functor, then there is a long exact sequence

$$
\cdots \to \mathbb{L}_n F(A) \to \mathbb{L}_n F(B) \to \mathbb{L}_n F(C) \to \mathbb{L}_{n-1} F(A)
\to \cdots \to \mathbb{L}_1 F(C) \to \mathbb{L}_0 F(A) \to \mathbb{L}_0 F(B) \to \mathbb{L}_0 F(C) \to 0.
$$

If $F$ is contravariant instead, there is a long exact sequence

$$
\cdots \leftarrow \mathbb{R}^n F(A) \leftarrow \mathbb{R}^n F(B) \leftarrow \mathbb{R}^n F(C) \leftarrow \mathbb{R}^{n-1} F(A)
\leftarrow \cdots \leftarrow \mathbb{R}^1 F(C) \leftarrow \mathbb{R}^0 F(A) \leftarrow \mathbb{R}^0 F(B) \leftarrow \mathbb{R}^0 F(C) \leftarrow 0.
$$

Proof. This follows from the third assertion of Proposition 44 together with the well-known long exact homology sequence for exact triangles in $\mathcal{D}e(\mathcal{C})$.

Lemma 47. Let $F : \mathcal{I} \to \mathcal{C}$ be a homological functor. The following assertions are equivalent:

1. $F$ is $\mathcal{I}$-exact;
2. $\mathbb{L}_0 F(A) \cong F(A)$ and $\mathbb{L}_p F(A) = 0$ for all $p > 0$, $A \in \mathcal{I}$;
3. $\mathbb{L}_0 F(A) \cong F(A)$ for all $A \in \mathcal{I}$.

The analogous assertions for contravariant functors are equivalent as well.

Proof. If $F$ is $\mathcal{I}$-exact, then $F$ maps $\mathcal{I}$-exact chain complexes in $\mathcal{I}$ to exact chain complexes in $\mathcal{C}$. This applies to $\mathcal{I}$-projective resolutions, so that (1)$\implies$(2)$\implies$(3). It follows from (3) and Lemma 46 that $F$ maps $\mathcal{I}$-epimorphisms to epimorphisms. Since this characterises $\mathcal{I}$-exact functors, we get $(3)\implies(1)$.

It can happen that $\mathbb{L}_p F = 0$ for all $p > 0$ although $F$ is not $\mathcal{I}$-exact.

We have a natural transformation $\mathbb{L}_0 F(A) \to F(A)$ (or $F(A) \to \mathbb{R}^0 F(A)$), which is induced by the augmentation map $P^\ast \to A$ for an $\mathcal{I}$-projective resolution. Lemma 47 shows that these maps are usually not bijective, although this happens frequently for derived functors on Abelian categories.

Definition 48. We let $\text{Ext}_{\mathcal{I}, \mathcal{J}}^n(A, B)$ be the $n$th right derived functor with respect to $\mathcal{J}$ of the contravariant functor $A \mapsto \mathcal{I}(A, B)$.

We have natural maps $\mathcal{I}(A, B) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(A, B)$, which usually are not invertible. Lemma 46 yields long exact sequences

$$
\cdots \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(A, D) \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^n(B, D) \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^n(C, D) \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^{n-1}(A, D) \leftarrow \cdots
\leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^1(C, D) \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^n(A, D) \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(B, D) \leftarrow \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(C, D) \leftarrow 0
$$

for any $\mathcal{I}$-exact exact triangle $A \to B \to C \to \Sigma A$ and any $D \in \mathcal{I}$.

We claim that there are similar long exact sequences

$$
0 \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(D, A) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(D, B) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^0(D, C) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^1(D, A) \to \cdots
\to \text{Ext}_{\mathcal{I}, \mathcal{J}}^{n-1}(D, C) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^n(D, A) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^n(D, B) \to \text{Ext}_{\mathcal{I}, \mathcal{J}}^n(D, C) \to \cdots
$$

in the second variable. Since $\mathcal{P}(D)_n$ is $\mathcal{I}$-projective, the sequences

$$
0 \to \mathcal{I}(\mathcal{P}(D)_n, A) \to \mathcal{I}(\mathcal{P}(D)_n, B) \to \mathcal{I}(\mathcal{P}(D)_n, C) \to 0
$$

are exact for all $n \in \mathbb{N}$. This extension of chain complexes yields the desired long exact sequence.
We list a few more elementary properties of derived functors. We only spell things out for the left derived functors $L_n F : \mathcal{I} \to \mathcal{C}$ of a covariant functor $F : \mathcal{I} \to \mathcal{C}$. Similar assertions hold for right derived functors of contravariant functors.

The derived functors $L_n F$ satisfy $I \subseteq \ker L_n F$ and hence descend to functors $L_n F : \mathcal{I}/I \to \mathcal{C}$ because the zero map $P(A) \to P(B)$ is a chain map lifting of $f$ if $f \in \mathcal{I}(A, B)$. As a consequence, $L_n F(A) \cong 0$ if $A$ is $I$-contractible. The long exact homology sequences of Lemma 46 show that $L_n F(f) : L_n F(A) \to L_n F(B)$ is invertible if $f \in \mathcal{I}(A, B)$ is an $I$-equivalence.

**Warning 49.** The derived functors $L_n F$ are not homological and therefore do not deserve to be called $I$-exact even though they vanish on $I$-phantom maps. Lemma 10 shows that these functors are only half-exact on $I$-exact triangles. Thus $L_n F(f)$ need not be monic (or epic) if $f$ is $I$-monic (or $I$-epic). The problem is that the $I$-projective resolution functor $P : \mathcal{I} \to \mathcal{I}(\mathcal{I})$ is not exact – it even fails to be stable.

The following remarks require a more advanced background in homological algebra and are not going to be used in the sequel.

**Remark 50.** The derived functors introduced above, especially the Ext functors, can be interpreted in terms of derived categories.

We have already observed in [3.2.1] that the $I$-exact chain complexes form a thick subcategory of $\mathcal{I}(\mathcal{I})$. The augmentation map $P(A) \to A$ of an $I$-projective resolution of $A \in \mathcal{I}$ is a quasi-isomorphism with respect to this thick subcategory. The chain complex $P(A)$ is projective (see [12]), that is, for any chain complex $C_\bullet$, the space of morphisms $A \to C_\bullet$ in the derived category $\mathcal{D}(\mathcal{I}, \mathcal{I})$ agrees with $[P(A), C_\bullet]$. Especially, $\Ext^n_{\mathcal{I}, I}(A, B)$ is the space of morphisms $A \to B[n]$ in $\mathcal{D}(\mathcal{I}, \mathcal{I})$.

Now let $F : \mathcal{I} \to \mathcal{C}$ be an additive covariant functor. Extend it to an exact functor $\tilde{F} : \mathcal{I}(\mathcal{I}) \to \mathcal{I}(\mathcal{C})$. It has a total left derived functor

$$L\tilde{F} : \mathcal{D}(\mathcal{I}, \mathcal{I}) \to \mathcal{D}(\mathcal{C}), \quad \tilde{F}(A) \mapsto L\tilde{F}(P(A)).$$

By definition, we have $L_n F(A) := H_n(L\tilde{F}(A))$.

**Remark 51.** In classical Abelian categories, the Ext groups form a graded ring, and the derived functors form graded modules over this graded ring. The same happens in our context. The most conceptual construction of these products uses the description of derived functors sketched in Remark 50.

Recall that we may view elements of $\Ext^n_{\mathcal{I}, I}(A, B)$ as morphisms $A \to B[n]$ in the derived category $\mathcal{D}(\mathcal{I}, \mathcal{I})$. Taking translations, we can also view them as morphisms $A[m] \to B[n+m]$ for any $m \in \mathbb{Z}$. The usual composition in the category $\mathcal{D}(\mathcal{I}, \mathcal{I})$ therefore yields an associative product

$$\Ext^n_{\mathcal{I}, I}(B, C) \otimes \Ext^m_{\mathcal{I}, I}(A, B) \to \Ext^{n+m}_{\mathcal{I}, I}(A, C).$$

Thus we get a graded additive category with morphism spaces $(\Ext^n_{\mathcal{I}, I}(A, B))_{n \in \mathbb{N}}$.

Similarly, if $F : \mathcal{I} \to \mathcal{C}$ is an additive functor and $L\tilde{F} : \mathcal{D}(\mathcal{I}, \mathcal{I}) \to \mathcal{D}(\mathcal{C})$ is as in Remark 51, then a morphism $A \to B[n]$ in $\mathcal{D}(\mathcal{I}, \mathcal{I})$ induces a morphism $L\tilde{F}(A) \to L\tilde{F}(B)[n]$ in $\mathcal{D}(\mathcal{C})$. Passing to homology, we get canonical maps

$$\Ext^n_{\mathcal{I}, I}(A, B) \to \text{Hom}_\mathcal{C}(L\tilde{F}_m(A), L\tilde{F}_{m-n}(B)) \quad \forall m \geq n,$$
which satisfy an appropriate associativity condition. For a contravariant functor, we get canonical maps
\[ \text{Ext}^n_{\mathcal{C}}(A, B) \to \text{Hom}_C(\mathbb{R}F^m(B), \mathbb{R}F^{m+n}(A)) \quad \forall m \geq 0. \]

### 3.6. Projective objects via adjointness

We develop a method for constructing enough projective objects. Let \( \mathcal{T} \) and \( \mathcal{C} \) be stable additive categories, let \( F: \mathcal{T} \to \mathcal{C} \) be a stable additive functor, and let \( \mathcal{I} := \ker F \). In our applications, \( \mathcal{T} \) is triangulated and the functor \( F \) is either exact or stable and homological.

Recall that a covariant functor \( R: \mathcal{T} \to \mathsf{Ab} \) is \textit{(co)representable} if it is naturally isomorphic to \( \mathcal{T}(A, -) \) for some \( A \in \mathcal{T} \), which is then unique. If the functor \( B \mapsto \mathcal{C}(A, F(B)) \) on \( \mathcal{T} \) is representable, we write \( \mathcal{F}^+(A) \) for the representing object. By construction, we have natural isomorphisms
\[ \mathcal{T}(\mathcal{F}^+(A), B) \cong \mathcal{C}(A, F(B)) \]
for all \( B \in \mathcal{T} \). Let \( \mathcal{C}' \) be the full subcategory of all objects \( A \in \mathcal{C} \) for which \( \mathcal{F}^+(A) \) is defined. Then \( \mathcal{F}^+ \) is a functor \( \mathcal{C}' \to \mathcal{T} \), which we call the \textit{(partially defined) left adjoint} of \( F \). Although one usually assumes \( \mathcal{C}' = \mathcal{C} \), we shall also need \( \mathcal{F}^+ \) in cases where it is not defined everywhere.

The functor \( B \mapsto \mathcal{C}(A, F(B)) \) for \( A \in \mathcal{C}' \) vanishes on \( \mathcal{I} = \ker F \) for trivial reasons. Hence \( \mathcal{F}^+(A) \in \mathcal{T} \) is \( \mathcal{I} \)-projective. This simple observation is surprisingly powerful: as we shall see, it often yields all \( \mathcal{I} \)-projective objects.

**Remark 52.** We have \( \mathcal{F}^+(\Sigma A) \cong \Sigma \mathcal{F}^+(A) \) for all \( A \in \mathcal{C}' \), so that \( \Sigma(\mathcal{C}') = \mathcal{C}' \). Moreover, \( \mathcal{F}^+ \) commutes with infinite direct sums (as far as they exist in \( \mathcal{T} \)) because
\[ \mathcal{T} \left( \bigoplus \mathcal{F}^+(A_i), B \right) \cong \prod \mathcal{T}(\mathcal{F}^+(A_i), B) \cong \prod \mathcal{C}(A_i, F(B)) \cong \mathcal{C} \left( \bigoplus A_i, F(B) \right). \]

**Example 53.** Consider the functor \( K_*: \mathsf{KK} \to \mathsf{Ab}^{Z/2} \). Let \( \mathcal{Z} \in \mathsf{Ab}^{Z/2} \) denote the trivially graded Abelian group \( \mathcal{Z} \). Notice that
\[
\text{Hom}(\mathcal{Z}, K_*(A)) \cong K_0(A) \cong \text{KK}(\mathcal{C}, A), \\
\text{Hom}(\mathcal{Z}[1], K_*(A)) \cong K_1(A) \cong \text{KK}(\mathcal{C}(\mathbb{R}), A),
\]
where \( \mathcal{Z}[1] \) means \( \mathcal{Z} \) in odd degree. Hence \( K_0^+(\mathcal{Z}) = \mathcal{C} \) and \( K_0^+(\mathcal{Z}[1]) = \mathcal{C}(\mathbb{R}) \). More generally, Remark 52 shows that \( K_0^+(A) \) is defined if both the even and odd parts of \( A \in \mathsf{Ab}^{Z/2} \) are countable free Abelian groups: it is a direct sum of at most countably many copies of \( \mathcal{C} \) and \( \mathcal{C}(\mathbb{R}) \). Hence all such countable direct sums are \( \mathcal{I}_K \)-projective (we briefly say \( K \)-projective). As we shall see, \( K_*^+ \) is not defined on all of \( \mathsf{Ab}^{Z/2} \); this is typical of homological functors.

**Example 54.** Consider the functor \( H_*: \mathcal{H}(\mathcal{C}; \mathbb{Z}/p) \to \mathcal{C}^{Z/p} \) of Example 9. Let \( j: \mathcal{C}^{Z/p} \to \mathcal{H}(\mathcal{C}; \mathbb{Z}/p) \) be the functor that views an object of \( \mathcal{C}^{Z/p} \) as a \( p \)-periodic chain complex whose boundary map vanishes.

A chain map \( j(A) \to B_* \) for \( A \in \mathcal{C}^{Z/p} \) and \( B_* \in \mathcal{H}(\mathcal{C}; \mathbb{Z}/p) \) is a family of maps \( \varphi_n: A_n \to \ker(d_n): B_n \to B_{n-1} \). Such a family is chain homotopic to 0 if and only if each \( \varphi_n \) lifts to a map \( A_n \to B_{n+1} \). Suppose that \( A_n \) is projective for all \( n \in \mathbb{Z}/p \). Then such a lifting exists if and only if \( \varphi_n(A_n) \subseteq d_{n+1}(B_{n+1}) \). Hence
\[ [j(A), B_*] \cong \prod_{n \in \mathbb{Z}/p} \mathcal{C}(A_n, H_n(B_*)) \cong \mathcal{C}^{Z/p}(A, H_*(B_*)). \]
As a result, the left adjoint of $H_*$ is defined on the subcategory of projective objects $\mathcal{C}^{\mathbb{Z}/p} \subseteq \mathcal{C}^{\mathbb{Z}/p}$ and we will show in Proposition 3.8 that all $H_*$-projective objects are of the form $H_*(A)$ for some $A \in \mathcal{C}^{\mathbb{Z}/p}$ (provided $\mathcal{C}$ has enough projective objects).

By duality, analogous results hold for injective objects: the domain of the right adjoint of $H_*$ is the subcategory of injective objects of $\mathcal{C}^{\mathbb{Z}/p}$, the right adjoint is equal to $j$ on this subcategory, and this provides all $H_*$-injective objects of $\mathcal{N}(\mathcal{C}, \mathbb{Z}/p)$.

These examples show that $F^+$ yields many ker $F$-projective objects. We want to get enough ker $F$-projective objects in this fashion, assuming that $F^+$ is defined on enough of $\mathcal{C}$. In order to treat ideals of the form $\bigcap F_i$, we now consider a more complicated setup. Let $\{\mathcal{C}_i \mid i \in I\}$ be a set of stable homological or triangulated categories together with full subcategories $\mathcal{V}\mathcal{C}_i \subseteq \mathcal{C}_i$ and stable homological or exact functors $F_i: \mathcal{I} \to \mathcal{C}_i$ for all $i \in I$. Assume that

- the left adjoint $F^+_i$ is defined on $\mathcal{V}\mathcal{C}_i$ for all $i \in I$;
- there is an epimorphism $P \to F_i(A)$ in $\mathcal{C}_i$ with $P \in \mathcal{V}\mathcal{C}_i$ for any $i \in I$, $A \in \mathcal{I}$;
- the set of functors $F^+_i: \mathcal{V}\mathcal{C}_i \to \mathcal{I}$ is cointegrable, that is, $\bigoplus_{i \in I} F^+_i(B_i)$ exists for all families of objects $B_i \in \mathcal{V}\mathcal{C}_i$, $i \in I$.

The reason for the notation $\mathcal{V}\mathcal{C}_i$ is that for a homological functor $F_i$ we usually take $\mathcal{V}\mathcal{C}_i$ to be the class of projective objects of $\mathcal{C}_i$; if $F_i$ is exact, then we often take $\mathcal{V}\mathcal{C}_i = \mathcal{C}_i$. But it may be useful to choose a smaller category, as long as it satisfies the second condition above.

**Proposition 55.** In this situation, there are enough $\mathcal{I}$-projective objects, and $\mathcal{V}\mathcal{I}$ is generated by $\bigcup_{i \in I} \{F^+_i(B) \mid B \in \mathcal{V}\mathcal{C}_i\}$. More precisely, an object of $\mathcal{I}$ is $\mathcal{I}$-projective if and only if it is a retract of $\bigoplus_{i \in I} F^+_i(B_i)$ for a family of objects $B_i \in \mathcal{V}\mathcal{C}_i$.

**Proof.** Let $\mathcal{V}\mathcal{I}_0 := \bigcup_{i \in I} \{F^+_i(B) \mid B \in \mathcal{V}\mathcal{C}_i\}$ and $\mathcal{V}\mathcal{I}_0 := (\mathcal{V}\mathcal{I}_0)_\mathcal{E}$. To begin with, we observe that any object of the form $F^+_i(B)$ with $B \in \mathcal{V}\mathcal{C}_i$ is ker $F_i$-projective and hence $\mathcal{I}$-projective because $\mathcal{I} \subseteq$ ker $F_i$. Hence $\mathcal{V}\mathcal{I}_0$ consists of $\mathcal{I}$-projective objects.

Let $A \in \mathcal{I}$. For each $i \in I$, there is an epimorphism $p_i: B_i \to F_i(A)$ with $B_i \in \mathcal{V}\mathcal{C}_i$. The direct sum $B := \bigoplus_{i \in I} F^+_i(B_i)$ exists. We have $B \in \mathcal{V}\mathcal{I}_0$ by construction. We are going to construct an $\mathcal{I}$-epimorphism $p: B \to A$. This shows that there are enough $\mathcal{I}$-projective objects.

The maps $p_i: B_i \to F_i(A)$ provide maps $\tilde{p}_i: F^+_i(B_i) \to A$ via the adjointness isomorphisms $\mathcal{I}(F^+_i(B_i), A) \cong \mathcal{C}_i(B_i, F_i(A))$. We let $p := \sum p_i: \bigoplus F^+_i(B_i) \to A$. We must check that $p$ is an $\mathcal{I}$-epimorphism. Equivalently, $p$ is ker $F_i$-epic for all $i \in I$: this is, in turn equivalent to $F_i(p)$ being an epimorphism in $\mathcal{C}_i$ for all $i \in I$, because of Lemma 22 or 25. This is what we are going to prove.

The identity map on $F^+_i(B_i)$ yields a map $\alpha_i: B_i \to F_i F^+_i(B_i)$ via the adjointness isomorphism $\mathcal{I}(F^+_i(B_i), F^+_i(B_i)) \cong \mathcal{C}_i(B_i, F_i F^+_i(B_i))$. Composing with the map

$$F_i F^+_i(B_i) \to F_i \left( \bigoplus F^+_i(B_i) \right) = F_i(B)$$

induced by the coordinate embedding $F^+_i(B_i) \to B$, we get a map $\alpha_i': B_i \to F_i(B)$.

The naturality of the adjointness isomorphisms yields $F_i(\tilde{p}_i) \circ \alpha_i = p_i$ and hence $F_i(p) \circ \alpha_i' = p_i$. The map $p_i$ is an epimorphism by assumption. Now we use a cancellation result for epimorphisms: if $f \circ g$ is an epimorphism, then so is $f$. Thus $F_i(p)$ is an epimorphism as desired.
If $A$ is $\mathcal{I}$-projective, then the $\mathcal{I}$-epimorphism $p \colon B \to A$ splits; to see this, embed $p$ in an exact triangle $N \to B \to A \to \Sigma N$ and observe that the map $A \to \Sigma N$ belongs to $\mathcal{I}(A, \Sigma N) = 0$. Therefore, $A$ is a retract of $B$. Since $\mathcal{I}_0$ is closed under retracts and $B \in \mathcal{I}_0$, we get $A \in \mathcal{I}_0$. Hence $\mathcal{I}_0$ generates all $\mathcal{I}$-projective objects. \qed

3.7. The universal exact homological functor. For the following results, it is essential to define an ideal by a single functor $F$ instead of a family of functors as in Proposition 55.

Definition 56. Let $\mathcal{I}$ be a homological ideal in a triangulated category $\mathcal{T}$. An $\mathcal{I}$-exact stable homological functor $F : \mathcal{T} \to \mathcal{C}$ is called universal if any other $\mathcal{I}$-exact stable homological functor $G : \mathcal{T} \to \mathcal{C}'$ factors as $G = G \circ F$ for a stable exact functor $G : \mathcal{C} \to \mathcal{C}'$ that is unique up to natural isomorphism.

This universal property characterises $F$ uniquely up to natural isomorphism. We have constructed such a functor in \cite{3.2.1}. Beligiannis constructs it in \cite{3} \S3 using a localisation of the Abelian category $\mathcal{Coh}(\mathcal{T})$ at a suitable Serre subcategory; he calls this functor projectivisation functor and its target category Steenrod category.

This notation is motivated by the special case of the Adams spectral sequence. The following theorem allows us to check whether a given functor is universal:

Theorem 57. Let $\mathcal{T}$ be a triangulated category, let $\mathcal{I} \subseteq \mathcal{T}$ be a homological ideal, and let $F : \mathcal{T} \to \mathcal{C}$ be an $\mathcal{I}$-exact stable homological functor into a stable Abelian category $\mathcal{C}$, let $\mathcal{P} \mathcal{C}$ be the class of projective objects in $\mathcal{C}$. Suppose that idempotent morphisms in $\mathcal{T}$ split.

The functor $F$ is the universal $\mathcal{I}$-exact stable homological functor and there are enough $\mathcal{I}$-projective objects in $\mathcal{T}$ if and only if

- $\mathcal{C}$ has enough projective objects;
- the adjoint functor $F^+ \text{ is defined on } \mathcal{P} \mathcal{C}$;
- $F \circ F^+(A) \cong A$ for all $A \in \mathcal{P} \mathcal{C}$.

Proof. Suppose first that $F$ is universal and that there are enough $\mathcal{I}$-projective objects. Then $F$ is equivalent to the projectivisation functor of $\mathcal{T}$. The various properties of this functor listed in \cite{4} Proposition 4.19 include the following:

- there are enough projective objects in $\mathcal{C}$;
- $F$ induces an equivalence of categories $\mathcal{P}_3 \cong \mathcal{P} \mathcal{C}$ ($\mathcal{P}_3$ is the class of projective objects in $\mathcal{T}$);
- $\mathcal{C}(F(A), F(B)) \cong \mathcal{T}(A, B)$ for all $A \in \mathcal{P}_3, B \in \mathcal{T}$.

Here we use the assumption that idempotents in $\mathcal{T}$ split. The last property is equivalent to $F^+ \circ F(A) \cong A$ for all $A \in \mathcal{P}_3$. Since $\mathcal{P}_3 \cong \mathcal{P} \mathcal{C}$ via $F$, this implies that $F^+$ is defined on $\mathcal{P} \mathcal{C}$ and that $F \circ F^+(A) \cong A$ for all $A \in \mathcal{P} \mathcal{C}$. Thus $F$ has the properties listed in the statement of the theorem.

Now suppose conversely that $F$ has these properties. Let $\mathcal{P}_3 \subseteq \mathcal{T}$ be the essential range of $F^+ : \mathcal{P} \mathcal{C} \to \mathcal{T}$. We claim that $\mathcal{P}_3$ is the class of all $\mathcal{I}$-projective objects in $\mathcal{T}$. Since $F \circ F^+$ is equivalent to the identity functor on $\mathcal{P} \mathcal{C}$ by assumption, $F|_\mathcal{P}_3$ and $F^+$ provide an equivalence of categories $\mathcal{P}_3 \cong \mathcal{P} \mathcal{C}$. Since $\mathcal{C}$ is assumed to have enough projectives, the hypotheses of Proposition 55 are satisfied. Hence there are enough $\mathcal{I}$-projective objects in $\mathcal{T}$, and any object of $\mathcal{P}_3$ is a retract of an object of $\mathcal{P}_3$. Idempotent morphisms in the category $\mathcal{P}_3 \cong \mathcal{P} \mathcal{C}$ split because $\mathcal{C}$ is Abelian and retracts of projective objects are again projective. Hence $\mathcal{P}_3$ is closed.
under retracts in $\mathbb{T}$, so that $\mathbb{P}' \cong \mathbb{P}$. It also follows that $F$ and $F^+$ provide an equivalence of categories $\mathbb{P}_3 \cong \mathbb{P}$. Hence $F^+ \circ F(A) \cong A$ for all $A \in \mathbb{P}_3$, so that we get $\mathbb{C}(F(A), F(B)) \cong \mathbb{T}(F^+ \circ F(A), B) \cong \mathbb{T}(A, B)$ for all $A \in \mathbb{P}_3$, $B \in \mathbb{T}$.

Now let $G: \mathbb{T} \to \mathbb{C}'$ be a stable homological functor. We will later assume $G$ to be $\mathbb{T}$-exact, but the first part of the following argument works in general. Since $F$ provides an equivalence of categories $\mathbb{P}_3 \cong \mathbb{P}$, the rule $G(F(P)) := G(P)$ defines a functor $G$ on $\mathbb{P}$. This yields a functor $\mathcal{H}(G): \mathcal{H}(\mathbb{P}) \to \mathcal{H}(\mathbb{C}')$. Since $\mathbb{C}$ has enough projective objects, the construction of projective resolutions provides a functor $P: \mathbb{C} \to \mathcal{H}(\mathbb{P})$. We let $G$ be the composite functor $G: \mathbb{C} \xrightarrow{P} \mathcal{H}(\mathbb{P}) \xrightarrow{\mathcal{H}(G)} \mathcal{H}(\mathbb{C}') \xrightarrow{\mathcal{H}_0} \mathbb{C}'$.

This functor is right-exact and satisfies $G \circ F = G$ on $\mathbb{T}$-projective objects of $\mathbb{T}$.

Now suppose that $G$ is $\mathbb{T}$-exact. Then we get $G \circ F = G$ for all objects of $\mathbb{T}$ because this holds for $\mathbb{T}$-projective objects. We claim that $G$ is exact. Let $A \in \mathbb{C}$. Since $\mathbb{C}$ has enough projective objects, we can find a projective resolution of $A$. We may assume this resolution to have the form $F(P_\bullet)$ with $P_\bullet \in \mathcal{H}(\mathbb{P}_3)$ because $F(\mathbb{P}_3) \cong \mathbb{P}$. Lemma 28 yields that $P_\bullet$ is $\mathbb{T}$-exact except in degree 0. Since $\mathbb{T} \subseteq \mathbb{C}$, the chain complex $P_\bullet$ is $G$-exact in positive degrees as well, so that $G(P_\bullet)$ is exact except in degree 0 by Lemma 28. As a consequence, $L_pG(A) = 0$ for all $p > 0$. We also have $L_0G(A) = G(A)$ by construction. Thus $G$ is exact.

As a result, $G$ factors as $G = G \circ F$ for an exact functor $G: \mathbb{C} \to \mathbb{C}'$. It is clear that $G$ is stable. Finally, since $\mathbb{C}$ has enough projective objects, a functor on $\mathbb{C}$ is determined up to natural equivalence by its restriction to projective objects. Therefore, our factorisation of $G$ is unique up to natural equivalence. Thus $F$ is the universal $\mathbb{T}$-exact functor.

Remark 58. Let $\mathbb{P}' \mathbb{C} \subseteq \mathbb{P}$ be some subcategory such that any object of $\mathbb{C}$ is a quotient of a direct sum of objects of $\mathbb{P}' \mathbb{C}$. Equivalently, $(\mathbb{P}' \mathbb{C})_0 \mathbb{C} = \mathbb{P} \mathbb{C}$. Theorem 57 remains valid if we only assume that $F^+$ is defined on $\mathbb{P}' \mathbb{C}$ and that $F \circ F^+(A) \cong A$ holds for $A \in \mathbb{P}' \mathbb{C}$ because both conditions are evidently hereditary for direct sums and retracts.

Theorem 59. In the situation of Theorem 57, the domain of the functor $F^+$ contains $\mathbb{P}$, and its essential range is $\mathbb{P}_3$. The functors $F$ and $F^+$ restrict to equivalences of categories $\mathbb{P}_3 \cong \mathbb{P}$ inverse to each other.

An object $A \in \mathbb{T}$ is $\mathbb{T}$-projective if and only if $F(A)$ is projective and $\mathbb{C}(F(A), F(B)) \cong \mathbb{T}(A, B)$ for all $B \in \mathbb{T}$; following Ross Street 24, we call such objects $F$-projective. We have $F(A) \in \mathbb{P}$ if and only if there is an $\mathbb{T}$-equivalence $P \to A$ with $P \in \mathbb{P}_3$.

The functors $F$ and $F^+$ induce bijections between isomorphism classes of projective resolutions of $F(A)$ in $\mathbb{C}$ and isomorphism classes of $\mathbb{T}$-projective resolutions of $A \in \mathbb{T}$ in $\mathbb{T}$.

If $G: \mathbb{T} \to \mathbb{C}'$ is any (stable) additive functor, then there is a unique right-exact (stable) functor $G: \mathbb{C} \to \mathbb{C}'$ such that $G \circ F(P) = G(P)$ for all $P \in \mathbb{P}_3$.

The left derived functors of $G$ with respect to $\mathbb{T}$ and of $G$ are related by natural isomorphisms $L_nG \circ F(A) = L_nG(A)$ for all $A \in \mathbb{T}$, $n \in \mathbb{N}$. There is a similar statement for cohomological functors, which specialises to natural isomorphisms

$$\text{Ext}_{\mathbb{T}}^n(A, B) \cong \text{Ext}_{\mathbb{C}}^n(F(A), F(B)).$$
Remark 60. The assumption that idempotents split is only needed to check that the universal \( \mathfrak{f} \)-exact functor has the properties listed in Theorem 57. The converse directions of Theorem 57 and Theorem 59 do not need this assumption.

If \( \mathfrak{f} \) has countable direct sums or countable direct products, then idempotents in \( \mathfrak{f} \) automatically split by [19, §1.3]. This covers categories such as \( \text{KK} \) because they have countable direct sums.

3.8. Derived functors in homological algebra. Now we study the kernel \( \mathfrak{f}_H \) of the homology functor \( H_\cdot : \mathfrak{H}(\mathfrak{C}, \mathbb{Z}/p) \to \mathfrak{C}^\mathbb{Z}/p \) introduced in Example 9. We get exactly the same statements if we replace the homotopy category by its derived category and study the kernel of \( \mathfrak{H}_\cdot : \mathfrak{D}(\mathfrak{C}, \mathbb{Z}/p) \to \mathfrak{C}^\mathbb{Z}/p \). We often abbreviate \( \mathfrak{H}_\cdot \) to \( \mathfrak{H} \) and speak of \( \mathfrak{H} \)-epimorphisms, \( \mathfrak{H} \)-exact chain complexes, \( \mathfrak{H} \)-projective resolutions, and so on. We denote the full subcategory of \( \mathfrak{H} \)-projective objects in \( \mathfrak{H}(\mathfrak{C}, \mathbb{Z}/p) \) by \( \mathfrak{P}_H \).
We assume that the underlying Abelian category $\mathcal{C}$ has enough projective objects. Then the same holds for $\mathcal{C}^{Z/p}$, and we have $\Phi(\mathcal{C}^{Z/p}) \cong (\Phi C)^{Z/p}$. That is, an object of $\mathcal{C}^{Z/p}$ is projective if and only if its homogeneous pieces are.

**Theorem 61.** The category $\mathcal{H}(\mathcal{C}; Z/p)$ has enough $\mathcal{H}$-projective objects, and the functor $H_* : \mathcal{H}(\mathcal{C}; Z/p) \to \mathcal{C}^{Z/p}$ is the universal $\mathcal{H}$-exact stable homological functor. Its restriction to $\mathcal{H}_H$ provides an equivalence of categories $\mathcal{H}_H \cong \Phi \mathcal{C}^{Z/p}$. More concretely, a chain complex in $\mathcal{H}(\mathcal{C}; Z/p)$ is $\mathcal{H}$-projective if and only if it is homotopy equivalent to one with vanishing boundary map and projective entries.

The functor $H_*$ maps isomorphism classes of $\mathcal{H}$-projective resolutions of a chain complex $A$ in $\mathcal{H}(\mathcal{C}; Z/p)$ bijectively to isomorphism classes of projective resolutions of $H_*(A)$ in $\mathcal{C}^{Z/p}$. We have

$$\text{Ext}_H^n(\mathcal{H}(\mathcal{C}; Z/p); \mathcal{H}(A, B)) \cong \text{Ext}_\mathcal{C}^n(H_*(A), H_*(B)).$$

Let $F : \mathcal{C} \to \mathcal{C}'$ be some covariant additive functor and define

$$\tilde{F} : \mathcal{H}(\mathcal{C}; Z/p) \to \mathcal{H}(\mathcal{C}'; Z/p)$$

by applying $F$ entrywise. Then $L_n \tilde{F}(A) \cong L_n F(H_*(A))$ for all $n \in \mathbb{N}$. Similarly, we have $R^n \tilde{F}(A) \cong R^n F(H_*(A))$ if $F$ is a contravariant functor.

**Proof.** The category $\mathcal{C}^{Z/p}$ has enough projective objects by assumption. We have already seen in Example 54 that $H'$ is defined on $\Phi \mathcal{C}^{Z/p}$; this functor is denoted by $j$ in Example 54. It is clear that $H_* \circ j(A) \cong A$ for all $A \in \mathcal{C}^{Z/p}$. Now Theorem 57 shows that $H_*$ is universal. We do not need idempotent morphisms in $\mathcal{H}(\mathcal{C}; Z/p)$ to split by Remark 60.

\[ \square \]

**Remark 62.** Since the universal $\mathcal{I}$-exact functor is essentially unique, the universality of $H_* : \mathcal{H}(\mathcal{C}; Z/p) \to \mathcal{C}^{Z/p}$ means that we can recover this functor and hence the stable Abelian category $\mathcal{C}^{Z/p}$ from the ideal $\mathcal{I}_H \subseteq \mathcal{H}(\mathcal{C}; Z/p)$. That is, the ideal $\mathcal{I}_H$ and the functor $H_* : \mathcal{H}(\mathcal{C}; Z/p) \to \mathcal{C}^{Z/p}$ contain exactly the same amount of information.

For instance, if we forget the precise category $\mathcal{C}$ by composing $H_*$ with some faithful functor $\mathcal{C} \to \mathcal{C}'$, then the resulting homology functor $\mathcal{H}(\mathcal{C}; Z/p) \to \mathcal{C}'$ still has kernel $\mathcal{I}_H$. We can recover $\mathcal{C}^{Z/p}$ by passing to the universal $\mathcal{I}$-exact functor.

We compare this with the situation for truncation structures (3.3). These cannot exist for periodic categories such as $\mathcal{H}(\mathcal{C}; Z/p)$ for $p \geq 1$. Given the standard truncation structure on $\mathcal{H}(\mathcal{C})$, we can recover the Abelian category $\mathcal{C}$ as its core; we also get back the homology functors $H_n : \mathcal{H}(\mathcal{C}) \to \mathcal{C}$ for all $n \in \mathbb{Z}$. Conversely, the functor $H_* : \mathcal{H}(\mathcal{C}) \to \mathcal{C}$ together with the grading on $\mathcal{C}$ determines the truncation structure. Hence the standard truncation structure on $\mathcal{H}(\mathcal{C})$ contains the same amount of information as the functor $H_* : \mathcal{H}(\mathcal{C}) \to \mathcal{C}$ together with the grading on $\mathcal{C}$.

4. **The plain Universal Coefficient Theorem**

Now we study the ideal $\mathcal{I}_K := \ker K_* \subseteq \mathcal{K}$ of Example 5. We complete our analysis of this example and explain the Universal Coefficient Theorem for $\mathcal{K}$ in our framework. We call $\mathcal{I}_K$-projective objects and $\mathcal{I}_K$-exact functors briefly $K$-projective and $K$-exact and let $\mathcal{P}_K$ be the class of $K$-projective objects in $\mathcal{K}$.
Let $\mathbb{Ab}_{c}^{Z/2} \subseteq \mathbb{Ab}^{Z/2}$ be the full subcategory of countable $\mathbb{Z}/2$-graded Abelian groups. Since the $K$-theory of a separable $C^*$-algebra is countable, we may view $K_*$ as a stable homological functor $K_* : \mathbb{KK} \to \mathbb{Ab}_{c}^{Z/2}$.

**Theorem 63.** There are enough $K$-projective objects in $\mathbb{KK}$, and the universal $K$-exact functor is $K_* : \mathbb{KK} \to \mathbb{Ab}_{c}^{Z/2}$. It restricts to an equivalence of categories between $\mathcal{P}_K$ and the full subcategory $\mathbb{Ab}_{c}^{Z/2} \subseteq \mathbb{Ab}^{Z/2}$ of $\mathbb{Z}/2$-graded countable free Abelian groups. A separable $C^*$-algebra belongs to $\mathcal{P}_K$ if and only if it is $KK$-equivalent to $\bigoplus_{i \in I_0} \mathbb{C} \oplus \bigoplus_{i \in I_1} C_0(\mathbb{R})$ where the sets $I_0, I_1$ are at most countable.

If $A \in \mathbb{KK}$, then $K_*$ maps isomorphism classes of $K$-projective resolutions of $A$ in $\mathcal{Z}$ bijectively to isomorphism classes of free resolutions of $K_*(A)$. We have

$$\text{Ext}_{\mathbb{KK}, \mathcal{K}}^n(A, B) \cong \begin{cases} \text{Hom}_{\mathbb{Ab}_{c}^{Z/2}}(K_*(A), K_*(B)) & \text{for } n = 0; \\ \text{Ext}^1_{\mathbb{Ab}_{c}^{Z/2}}(K_*(A), K_*(B)) & \text{for } n = 1; \\ 0 & \text{for } n \geq 2. \end{cases}$$

Let $F : \mathbb{KK} \to \mathcal{C}$ be some covariant additive functor; then there is a unique right-exact functor $\tilde{F} : \mathbb{Ab}_{c}^{Z/2} \to \mathcal{C}$ with $F \circ K_* = \tilde{F}$. We have $\mathbb{L}_n F = (\mathbb{L}_n \tilde{F}) \circ K_*$ for all $n \in \mathbb{N}$; this vanishes for $n \geq 2$. Similar assertions hold for contravariant functors.

**Proof.** Notice that $\mathbb{Ab}_{c}^{Z/2} \subseteq \mathbb{Ab}^{Z/2}$ is an Abelian category. We shall denote objects of $\mathbb{Ab}_{c}^{Z/2}$ by pairs $(A_0, A_1)$ of Abelian groups. By definition, $(A_0, A_1) \in \mathbb{Ab}_{c}^{Z/2}$ if and only if $A_0$ and $A_1$ are countable free Abelian groups, that is, they are of the form $A_0 = \mathbb{Z}[I_0]$ and $A_1 = \mathbb{Z}[I_1]$ for at most countable sets $I_0, I_1$. It is well-known that any Abelian group is a quotient of a free Abelian group and that subgroups of free Abelian groups are again free. Moreover, free Abelian groups are projective. Hence $\mathbb{Ab}_{c}^{Z/2}$ is the subcategory of projective objects in $\mathbb{Ab}_{c}^{Z/2}$ and any object $G \in \mathbb{Ab}_{c}^{Z/2}$ has a projective resolution of the form $0 \to F_1 \to F_0 \to G$ with $F_0, F_1 \in \mathbb{Ab}_{c}^{Z/2}$. This implies that derived functors on $\mathbb{Ab}_{c}^{Z/2}$ only occur in dimensions $1$ and $0$.

As in Example 53, we see that $K_*^c$ is defined on $\mathbb{Ab}_{c}^{Z/2}$ and satisfies

$$K_*^c(\mathbb{Z}[I_0], \mathbb{Z}[I_1]) \cong \bigoplus_{i \in I_0} \mathbb{C} \oplus \bigoplus_{i \in I_1} C_0(\mathbb{R})$$

if $I_0, I_1$ are countable. We also have $K_* \circ K_*^c(\mathbb{Z}[I_0], \mathbb{Z}[I_1]) \cong (\mathbb{Z}[I_0], \mathbb{Z}[I_1])$, so that the hypotheses of Theorem 57 are satisfied. Hence there are enough $K$-projective objects and $K_*$ is universal. The remaining assertions follow from Theorem 59 and our detailed knowledge of the homological algebra in $\mathbb{Ab}_{c}^{Z/2}$.

**Example 64.** Consider the stable homological functor

$$F : \mathbb{KK} \to \mathbb{Ab}_{c}^{Z/2}, \quad A \mapsto K_*(A \otimes B)$$

for some $B \in \mathbb{KK}$, where $\otimes$ denotes, say, the spatial $C^*$-tensor product. We claim that the associated right-exact functor $\mathbb{Ab}_{c}^{Z/2} \to \mathbb{Ab}_{c}^{Z/2}$ is

$$\tilde{F} : \mathbb{Ab}_{c}^{Z/2} \to \mathbb{Ab}_{c}^{Z/2}, \quad G \mapsto G \otimes K_*(B).$$

It is easy to check $F \circ K_*^c(G) \cong G \otimes K_*(B) \cong \tilde{F}(G)$ for $G \in \mathbb{Ab}_{c}^{Z/2}$. Since the functor $G \mapsto G \otimes K_*(B)$ is right-exact and agrees with $\tilde{F}$ on projective objects, we
get \( \tilde{F}(G) \cong G \otimes K_\ast(B) \) for all \( G \in \mathfrak{Ab}_r^{2/2} \). Hence the derived functors of \( F \) are

\[
L_n F(A) \cong \begin{cases} 
K_\ast(A) \otimes K_\ast(B) & \text{for } n = 0; \\
\text{Tor}^1(K_\ast(A), K_\ast(B)) & \text{for } n = 1; \\
0 & \text{for } n \geq 2.
\end{cases}
\]

Here we use the same graded version of Tor as in the Künneth Theorem (5).

**Example 65.** Consider the stable homological functor

\[ F : KK \to \mathfrak{Ab}_r^{2/2}, \quad B \mapsto KK_\ast(A, B) \]

for some \( A \in KK \). We suppose that \( A \) is a *compact* object of \( KK \), that is, the functor \( F \) commutes with direct sums. Then \( KK_\ast(A, K_\ast(G)) \cong KK_\ast(A, \mathcal{C}) \otimes G \) for all \( G \in \mathfrak{Ab}_r^{2/2} \) because this holds for \( G = (\mathbb{Z}, 0) \) and is inherited by suspensions and direct sums. Now we get \( \tilde{F}(G) \cong KK_\ast(A, \mathcal{C}) \otimes G \) for all \( G \in \mathfrak{Ab}_r^{2/2} \) as in Example 64. Therefore,

\[
L_n F(B) \cong \begin{cases} 
KK_\ast(A, \mathcal{C}) \otimes K_\ast(B) & \text{for } n = 0; \\
\text{Tor}^1(KK_\ast(A, \mathcal{C}), K_\ast(B)) & \text{for } n = 1; \\
0 & \text{for } n \geq 2.
\end{cases}
\]

Generalizing Examples 64 and 65, we have \( \tilde{F}(G) \cong F(\mathcal{C}) \otimes G \) and hence

\[
\mathbb{L}_n F(B) \cong \begin{cases} 
F(\mathcal{C}) \otimes K_\ast(B) & \text{for } n = 0, \\
\text{Tor}^1(F(\mathcal{C}), K_\ast(B)) & \text{for } n = 1, \\
0 & \text{for } n \geq 2,
\end{cases}
\]

for any covariant functor \( F : KK \to \mathcal{C} \) that commutes with direct sums.

Similarly, if \( F : KK \to \mathcal{C} \) is contravariant and maps direct sums to direct products, then \( \tilde{F}(G) \cong \text{Hom}(G, F(\mathcal{C})) \) and

\[
\mathbb{R}^n F(B) \cong \begin{cases} 
\text{Hom}(K_\ast(B), F(\mathcal{C})) & \text{for } n = 0, \\
\text{Ext}^1(K_\ast(B), F(\mathcal{C})) & \text{for } n = 1, \\
0 & \text{for } n \geq 2.
\end{cases}
\]

The description of \( \text{Ext}^n_{KK, \mathfrak{C}} \) in Theorem 63 is a special case of this.

**4.1. Universal Coefficient Theorem in the hereditary case.** In general, we need spectral sequences in order to relate the derived functors \( \mathbb{L}_n F \) back to \( F \). We shall discuss this in [16]. Here we only treat the simple case where we have projective resolutions of length 1. The following universal coefficient theorem is very similar to but slightly more general than [4, Theorem 4.27] because we do not require \( \mathfrak{J} \)-equivalences to be invertible.

**Theorem 66.** Let \( \mathfrak{J} \) be a homological ideal in a triangulated category \( \mathfrak{T} \). Let \( A \in \mathfrak{T} \) have an \( \mathfrak{J} \)-projective resolution of length 1. Suppose also that \( \mathfrak{T}(A, B) = 0 \) for all \( \mathfrak{J} \)-contractible \( B \). Let \( F : \mathfrak{T} \to \mathcal{C} \) be a homological functor, \( \tilde{F} : \mathfrak{T}^{\text{op}} \to \mathcal{C} \) a cohomological functor, and \( B \in \mathfrak{T} \). Then there are natural short exact sequences

\[
0 \to \mathbb{L}_n F_\ast(A) \to F_\ast(A) \to \mathbb{L}_1 F_{\ast-1}(A) \to 0,
\]

\[
0 \to \mathbb{R}^1 \tilde{F}^{\ast-1}(A) \to \tilde{F}^\ast(A) \to \mathbb{R}^0 \tilde{F}^\ast(A) \to 0,
\]

\[
0 \to \text{Ext}^1_{\mathfrak{T}, \mathfrak{J}}(\Sigma A, B) \to \mathfrak{T}(A, B) \to \text{Ext}^0_{\mathfrak{T}, \mathfrak{J}}(A, B) \to 0.
\]
Example 67. For the ideal $\mathfrak{I}_K \subseteq \text{KK}$, any object has a $K$-projective resolution of length 1 by Theorem $63$. The other hypothesis of Theorem $66$ holds if and only if $A$ satisfies the Universal Coefficient Theorem (UCT). The UCT for $\text{KK}(A, B)$ predicts $\text{KK}(A, B) = 0$ if $K_*(B) = 0$. Conversely, if this is the case, then Theorem $66$ applies, and our description of Ext$_{\text{KK}, 3K}$ in Theorem $63$ yields the UCT for $\text{KK}(A, B)$ for all $B$. This yields our claim.

Thus the UCT for $\text{KK}(A, B)$ is a special of Theorem $66$. In the situations of Examples $64$ and $65$ we get the familiar Künneth Theorems for $K_*(- \otimes B)$ and $\text{KK}_*(A, B)$. These arguments are very similar to the original proofs (see [5]). Our machinery allows us to treat other situations in a similar fashion.

Proof of Theorem $66$. We only write down the proof for homological functors. The cohomological case is dual and contains $\mathfrak{I}(-, B)$ as a special case.

Let $0 \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A$ be an $3$-projective resolution of length 1 and view it as an $3$-exact chain complex of length 3. Lemma $31$ yields a commuting diagram

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\delta_0} & \tilde{A} \\
| & & | & & | \\
P_1 & \xrightarrow{\delta_1} & P_0 & \xrightarrow{\delta_0} & A \\
\end{array}
\]

such that the top row is part of an $3$-exact exact triangle $P_1 \to P_0 \to \tilde{A} \to \Sigma P_1$ and $\alpha$ is an $3$-equivalence. We claim that $\alpha$ is an isomorphism in $\mathfrak{I}$.

We embed $\alpha$ in an exact triangle $\Sigma^{-1}B \to \tilde{A} \xrightarrow{\alpha} A \xrightarrow{\beta} B$. Lemma $24$ shows that $B$ is $3$-contractible because $\alpha$ is an $3$-equivalence. Hence $\mathfrak{I}(A, B) = 0$ by our assumption on $A$. This forces $\beta = 0$, so that our exact triangle splits: $\tilde{A} \cong A \oplus \Sigma^{-1}B$. Now we apply the functor $\mathfrak{I}(-, B)$ to the exact triangle $P_0 \to P_1 \to \tilde{A}$. The resulting long exact sequence has the form

\[
\cdots \leftarrow \mathfrak{I}(P_0, B) \leftarrow \mathfrak{I}(\tilde{A}, B) \leftarrow \mathfrak{I}(\Sigma P_1, B) \leftarrow \cdots.
\]

Since both $P_0$ and $P_1$ are $3$-projective and $B$ is $3$-contractible, we get $\mathfrak{I}(\tilde{A}, B) = 0$. Then $\mathfrak{I}(B, B) \subseteq \mathfrak{I}(\tilde{A}, B)$ vanishes as well, so that $B \cong 0$ and $\alpha$ is invertible.

We get an exact triangle in $\mathfrak{I}$ of the form $P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A \to \Sigma P_1$ because any triangle isomorphic to an exact one is itself exact.

Now we apply $F$. Since $F$ is homological, we get a long exact sequence

\[
\cdots \to F_*(P_1) \xrightarrow{F_*(\delta_1)} F_*(P_0) \to F_*(A) \to F_{*-1}(P_1) \xrightarrow{F_{*-1}(\delta_1)} F_{*-1}(P_0) \to \cdots.
\]

We cut this into short exact sequences of the form

\[
\text{coker}(F_*(\delta_1)) \to F_*(A) \to \ker(F_{*-1}(\delta_1)).
\]

Since $\text{coker} F_*(\delta_1) = L_0 F_*(A)$ and $\ker F_*(\delta_1) = L_1 F_*(A)$, we get the desired exact sequence. The map $L_0 F_*(A) \to F_*(A)$ is the canonical map induced by $\delta_0$. The other map $F_*(A) \to L_1 F_{*-1}(A)$ is natural for all morphisms between objects with an $3$-projective resolution of length 1 by Proposition $14$. \hfill \Box

The proof shows that in the situation of Theorem $66$ we have

\[
\text{Ext}^0_{\mathfrak{I}, 3}(A, B) \cong \mathfrak{I}/\mathfrak{I}(A, B), \quad \text{Ext}^1_{\mathfrak{I}, 3}(A, B) \cong \mathfrak{I}(A, \Sigma B).
\]
More generally, we can construct a natural map $\mathcal{I}(A, \Sigma B) \to \text{Ext}^1_{\mathcal{I}, \mathcal{J}}(A, B)$ for any homological ideal, using the $\mathcal{J}$-universal homological functor $F: \mathcal{I} \to \mathcal{E}$. We embed $f \in \mathcal{J}(A, \Sigma B)$ in an exact triangle $B \to C \to A \to \Sigma B$. We get an extension

$$[F(B) \to F(C) \to F(A)] \in \text{Ext}^1_{\mathcal{J}}(F(A), F(B))$$

because this triangle is $\mathcal{J}$-exact. This class $\kappa(f)$ in $\text{Ext}^1_{\mathcal{J}}(F(A), F(B))$ does not depend on auxiliary choices because the exact triangle $B \to C \to A \to \Sigma B$ is unique up to isomorphism. Theorem 59 yields $\text{Ext}^1_{\mathcal{J}, \mathcal{J}}(A, B) \cong \text{Ext}^1_{\mathcal{J}}(F(A), F(B))$ because $F$ is universal. Hence we get a natural map

$$\kappa: \mathcal{J}(A, \Sigma B) \to \text{Ext}^1_{\mathcal{I}, \mathcal{J}}(A, B).$$

We may view $\kappa$ as a secondary invariant generated by the canonical map

$$\mathcal{J}(A, B) \to \text{Ext}^0_{\mathcal{I}, \mathcal{J}}(A, B).$$

For the ideal $\mathcal{J}_K$, we get the same map $\kappa$ as in Example 5.

An Abelian category with enough projective objects is called hereditary if any subobject of a projective object is again projective. Equivalently, any object has a projective resolution of length 1. This motivates the following definition:

**Definition 68.** A homological ideal $\mathcal{I}$ in a triangulated category $\mathcal{T}$ is called hereditary if any object of $\mathcal{T}$ has a projective resolution of length 1.

If $\mathcal{J}$ is hereditary and if $\mathcal{J}$-equivalences are invertible, then Theorem 66 applies to all $A \in \mathcal{T}$ (and vice versa).

5. Crossed products for compact quantum groups

Let $G$ be a compact (quantum) group and write $C(G)$ and $C^*(G)$ for the Hopf $C^*$-algebras of $G$ and $G$. We study the homological algebra in $\text{KK}^G$ generated by the ideals $\mathcal{J}_G, \mathcal{J}_K \subseteq \mathcal{J}_\infty \subseteq \text{KK}$ defined in Example 7. Recall that $\mathcal{J}_K$ is the kernel of the crossed product functor

$$G \rtimes \omega: \text{KK}^G \to \text{KK}, \quad A \mapsto G \rtimes A,$$

whereas $\mathcal{J}_G$ is the kernel of the composite functor $K_* \circ (G \rtimes \omega)$. Since we have already analysed $K_*$ in 4, we can treat both ideals in a parallel fashion.

Our setup contains two classical special cases. First, $G$ may be a compact Lie group. Then $C(G)$ and $C^*(G)$ have the usual meaning, and the objects of $\text{KK}^G$ are separable $C^*$-algebras with a continuous action of $G$. Secondly, $C(G)$ may be the dual $C^*_\text{red}(H)$ of a discrete group $H$, so that $C^*(G) = C_0(H)$. Then the objects of $\text{KK}^G$ are separable $C^*$-algebras equipped with a (reduced) coaction of $H$. (We disregard the nuances between reduced and full coactions.) If we identify $\text{KK}^G_{C^*_\text{red}}(H) \cong \text{KK}^H$ using Baaj–Skandalis duality, then the crossed product functor $G \rtimes \omega$ corresponds to the forgetful functor $\text{KK}^H \to \text{KK}$.

Let $\mathcal{P}_K$ and $\mathcal{P}_K,\mathcal{J}_K$ be the classes of projective objects for the ideals $\mathcal{J}_K$ and $\mathcal{J}_K,\mathcal{J}_K$. Our first task is to find enough projective objects for these two ideals.

Let $\tau: \text{KK} \to \text{KK}^G$ be the functor that equips $B \in \text{KK}$ with the trivial $G$-action (that is, coaction of $C(G)$). This corresponds to the induction functor from the trivial quantum group to $C^*(G)$ under the equivalence $\text{KK}^G \cong \text{KK}_{C^*(G)}^G$.

The functors $\tau$ and $G \rtimes \omega$ are adjoint, that is, there are natural isomorphisms

\begin{equation}
\text{KK}^G(\tau(A), B) \cong \text{KK}(A, G \rtimes B)
\end{equation}
for all $A \in \text{KK}$, $B \in \text{KK}^G$. This generalisation of the Green–Julg Theorem is proved in [26 Théorème 5.10].

**Lemma 69.** There are enough projective objects for $\mathcal{I}_\kappa$ and $\mathcal{I}_{\kappa, K}$. We have

$$\mathfrak{P}_\kappa = (\tau(\text{KK}))_{[]} \quad \text{and} \quad \mathfrak{P}_{\kappa, K} = (\tau\mathbb{C}, \tau\mathcal{C}_0(\mathbb{R}))_{[]}.$$

**Proof.** The adjoint of $G \times \omega$ is defined on all of $\text{KK}$, which is certainly enough for Proposition 55. This yields the assertions for $\mathcal{I}_\kappa$ and $\mathfrak{P}_\kappa$; even more, $\mathfrak{P}_\kappa$ is the closure of $\tau(\text{KK})$ under retracts.

The adjoint of $K_\ast \circ (G \times \omega)$ is $\tau \circ K_\ast$, which is defined for free countable $\mathbb{Z}/2$-graded Abelian groups. Explicitly,

$$\tau \circ K_\ast (\mathbb{Z}[I_0], \mathbb{Z}[I_1]) \cong \bigoplus_{i \in I_0} \tau(\mathbb{C}) \oplus \bigoplus_{i \in I_1} \tau(\mathbb{C}_0(\mathbb{R})).$$

Since any object of $\mathfrak{A}^{\mathbb{Z}/2}_C$ is a quotient of one in $\mathfrak{A}^{\mathbb{Z}/2}_C$, Proposition 55 applies and yields the assertions about $\mathcal{I}_{\kappa, K}$ and $\mathfrak{P}_{\kappa, K}$; even more, $\mathfrak{P}_{\kappa, K}$ is the closure of $\tau \circ K_\ast (\mathfrak{A}^{\mathbb{Z}/2}_C)$ under retracts. $\square$

To proceed further, we describe the universal $\mathcal{I}$-exact functors. The functors $(G \times \omega)$ and $K_\ast \circ (G \times \omega)$ fail the criterion of Theorem 57 (unless $G = \{1\}$) because $G \times \tau(A) \cong C^*(G) \otimes A \not\cong A$.

This is not surprising because $G \times \omega$ is equivalent to a forgetful functor. The universal functor recovers a linearisation of the forgotten structure.

First we consider the ideal $\mathcal{I}_{\kappa, K}$. By Lemma 69, the objects $\tau(\mathbb{C})$ and $\Sigma \tau(\mathbb{C})$ generate all $\mathcal{I}_{\kappa, K}$-projective objects. Their internal symmetries are encoded by the $\mathbb{Z}/2$-graded ring $KK^G_\ast (\tau\mathbb{C}, \tau\mathcal{C})_{\text{op}}$; the superscript $\text{op}$ denotes that we take the product in reversed order. For classical compact groups, this ring is commutative, so that the order of multiplication does not matter; in general, the reversed-order product is the more standard choice. The following fact is well-known:

**Lemma 70.** The ring $KK^G_\ast (\tau\mathbb{C}, \tau\mathcal{C})_{\text{op}}$ is isomorphic to the representation ring $\text{Rep}(G)$ of $G$ for $* = 0$ and zero for $* = 1$.

We may take this as the definition of $\text{Rep}(G)$.

**Proof.** The adjointness isomorphism (5.1) identifies

$$KK^G_\ast (\tau\mathbb{C}, \tau\mathcal{C}) \cong KK_\ast (\mathcal{C}, G \times \tau\mathbb{C}) = KK_\ast (\mathcal{C}, C^*(G)) \cong K_\ast (C^*G).$$

Since $G$ is compact, $C^*G$ is a direct sum of matrix algebras, one for each irreducible representation of $G$. Hence the underlying Abelian group of $\text{Rep}(G)$ is the free Abelian group $\mathbb{Z}[^{\mathcal{C}}]_G$ on the set $\mathcal{C}$ of irreducible representations of $G$. The ring structure on $\text{Rep}(G)$ comes from the internal tensor product of representations: represent two elements $\alpha, \beta$ of $KK^G_\ast (\tau\mathbb{C}, \tau\mathcal{C})$ by differences of finite-dimensional representations $\tau, \varrho$ of $G$, then $\alpha \circ \beta \in KK^G_\ast (\tau\mathbb{C}, \tau\mathcal{C})$ is represented by $\varrho \otimes \tau$ because the product in $KK^G$ boils down to an exterior tensor product in this case (with reversed order). $\square$

**Example 71.** If $C(G) = C^*_{\text{red}}(H)$ for a discrete group $H$, then $\hat{G} = H$ and the product on $\text{Rep}(G) = \mathbb{Z}[H]$ is the usual convolution. Thus $\text{Rep}(G)$ is the group ring of $H$. 


If $G$ is a compact group, then $\text{Rep}(G)$ is the representation ring of $G$ in the usual sense.

For any $B \in \text{KK}^G$, the Kasparov product turns $\text{KK}^G(\tau\mathbb{C}, B)$ into a left module over the ring $\text{KK}^G(\tau\mathbb{C}, \tau\mathbb{C})^{\text{op}} \cong \text{Rep}(G)$. Thus $\text{KK}^G(\tau\mathbb{C}, B)$ becomes an object of the Abelian category $\text{Mod}(\text{Rep}(G)^{\mathbb{Z}/2})$ of $\mathbb{Z}/2$-graded countable $\text{Rep}(G)$-modules. We get a stable homological functor

$$F_K: \text{KK}^G \to \text{Mod}(\text{Rep}(G)^{\mathbb{Z}/2}), \quad F_K(B) := \text{KK}^G(\tau\mathbb{C}, B).$$

By [5,1], the underlying Abelian group of $\text{KK}^G(\tau\mathbb{C}, B)$ is

$$\text{KK}^G_*((\tau\mathbb{C}, B) \cong \text{KK}_*(\mathbb{C}, G \ltimes B) \cong \text{KK}_*(G \ltimes B).$$

Hence we still have $\ker F_K = \mathcal{I}_{K,K}$. We often write $F_K(B) = \text{K}_*(G \ltimes B)$ if it is clear from the context that we view $\text{K}_*(G \ltimes B)$ as an object of $\text{Mod}(\text{Rep}(G)^{\mathbb{Z}/2})$.

**Theorem 72.** The functor $F_K$ is the universal $\mathcal{I}_{K,K}$-exact functor.

The subcategory of $\mathcal{I}_{K,K}$-projective objects in $\text{KK}^G$ is equivalent to the subcategory of $\mathbb{Z}/2$-graded countable projective $\text{Rep}(G)$-modules. If $A \in \text{KK}^G$, then $F_K$ induces a bijection between isomorphism classes of $\mathcal{I}_{K,K}$-projective resolutions of $A$ and projective resolutions of $F_K(A)$ in $\text{Mod}(\text{Rep}(G)^{\mathbb{Z}/2})$.

If $A, B \in \text{KK}^G$, then

$$\text{Ext}^n_{\text{KK}^G,\mathcal{I}_{K,K}}(A, B) \cong \text{Ext}^n_{\text{Rep}(G)}(\text{K}_*(G \ltimes A), \text{K}_*(G \ltimes B)).$$

If $H: \text{KK}^G \to \mathcal{C}$ is homological and commutes with countable direct sums, then

$$L^nH(A) \cong \text{Tor}^n_{\text{Rep}(G)}(H_*(\tau\mathbb{C}), \text{K}_*(G \ltimes A));$$

if $H: (\text{KK}^G)^{\text{op}} \to \mathcal{C}$ is cohomological and turns countable direct sums into direct products, then

$$R^nH(A) \cong \text{Ext}^n_{\text{Rep}(G)}(\text{K}_*(G \ltimes A), H^*(\tau\mathbb{C}));$$

Here we use the right or left $\text{Rep}(G)$-module structure on $H_*(\tau\mathbb{C})$ that comes from the functoriality of $H$.

**Proof.** We verify universality using Theorem [57]. The category $\text{Mod}(\text{Rep}(G)^{\mathbb{Z}/2})$ has enough projective objects: countable free modules are projective, and any object is a quotient of a countable free module.

Given a free module $(\text{Rep}(G)[I_0], \text{Rep}(G)[I_1])$, we have natural isomorphisms

$$\text{Hom}_{\text{Rep}(G)}((\text{Rep}(G)[I_0], \text{Rep}(G)[I_1]), F_K(B)) \cong \prod_{\varepsilon \in \{0,1\}, i \in I_\varepsilon} \text{K}_*(G \ltimes B)$$

$$\cong \text{KK}^G\left(\bigoplus_{\varepsilon \in \{0,1\}, i \in I_\varepsilon} \Sigma^\varepsilon \tau\mathbb{C}, B \right).$$

Hence the adjoint functor $F_K^+$ is defined on countable free modules. Idempotents in $\text{KK}^G$ split by Remark [60]. Therefore, the domain of $F_K^+$ is closed under retracts and contains all projective objects of $\text{Mod}(\text{Rep}(G)^{\mathbb{Z}/2})$. It is easy to see that $F_K \circ F_K^+(A) \cong A$ for free modules. This extends to retracts and hence holds for all projective modules (compare Remark [58]). Now Theorem [57] yields that $F_K$ is universal.

The assertions about projective objects, projective resolutions, and Ext now follow from Theorem [59]. Theorem [59] also yields a formula for left derived functors in
terms of the right-exact functor $\tilde{H}: \mathcal{M}od(\text{Rep}G)_{\mathbb{Z}/2} \to \mathcal{C}$ associated to a homological functor $H: \text{KK}G \to \mathcal{C}$. It remains to compute $\tilde{H}$.

First we define the Tor objects in the statement of the theorem if $\mathcal{C}$ is the category of Abelian groups. Then $H_*(\tau\mathbb{C}) \in \mathcal{M}od(\text{Rep}G)^{\mathbb{Z}/2}$, and we can take the derived functors of the usual $\mathbb{Z}/2$-graded balanced tensor product $\otimes_{\text{Rep}G}$ for $\text{Rep}(G)$-modules. We claim that there are natural isomorphisms

$$\tilde{H}_*(M) \cong H_*(\tau\mathbb{C}) \otimes_{\text{Rep}G} M$$

for all $M \in \mathcal{M}od(\text{Rep}G)_{\mathbb{Z}/2}$. This holds for $M = \text{Rep}G$ and hence for all free modules because we have natural isomorphisms

$$\tilde{H}_*(\text{Rep}G) \cong H_*(F^*_K(\text{Rep}G)) \cong H_*(\tau\mathbb{C}) \cong H_*(\tau\mathbb{C}) \otimes_{\text{Rep}G} \text{Rep}G.$$

For general $M$, the functor $\tilde{H}(M)$ is computed by a free resolution because it is right-exact. Using this, one extends the computation to all modules. By definition, $\text{Tor}^n_{\text{Rep}G}(N, M)$ for $N \in \mathcal{M}od((\text{Rep}G)^{\mathbb{Z}/2})$, $M \in \mathcal{M}od(\text{Rep}G)_{\mathbb{Z}/2}$, is the $n$th left derived functor of the functor $N \otimes_{\text{Rep}G} -$ on $\mathcal{M}od(\text{Rep}G)_{\mathbb{Z}/2}$.

Now Theorem [22] yields the formula for $L_nH$ provided $H$ takes values in Abelian groups. The same argument works in general, we only need more complicated categories.

Let $\mathcal{C}^{\mathbb{Z}/2}[\text{Rep}(G)^{\mathbb{Z}/2}]$ be the category of $\mathbb{Z}/2$-graded objects $A$ of $\mathcal{C}$ together with a ring homomorphism $\text{Rep}(G)^{\mathbb{Z}/2} \to \mathcal{C}^{\mathbb{Z}/2}(A, A)$. We can extend the definition of $\otimes_{\text{Rep}G}$ to get an additive stable bifunctor

$$\otimes_{\text{Rep}G}: \mathcal{C}^{\mathbb{Z}/2}[\text{Rep}(G)^{\mathbb{Z}/2}] \otimes \mathcal{M}od(\text{Rep}G)_{\mathbb{Z}/2} \to \mathcal{C}^{\mathbb{Z}/2}.$$

Its derived functors are $\text{Tor}^n_{\text{Rep}G}$. As above, we see that this yields the derived functors of $H$.

The case of cohomological functors is similar and left to the reader. \qed

If $G$ is a compact group, then the same derived functors appear in the Universal Coefficient Theorem for $\text{KK}G$ by Jonathan Rosenberg and Claude Schochet ([22]). This is no coincidence, of course. It is explained in [16] how a homological ideal in a triangulated category generates a spectral sequence. This machinery applied to the ideal $J_{\kappa, K}$ yields the spectral sequence of [22].

In order to get the universal functor for the ideal $J_{\kappa}$, we must lift the $\text{Rep}(G)$-module structure on $\mathcal{K}_*(G \times B)$ to $G \times B$. Given any additive category $\mathcal{C}$, we define a category $\mathcal{C}[\text{Rep}G]$ as in the proof of Theorem [72] its objects are pairs $(A, \mu)$ where $A \in \mathcal{C}$ and $\mu$ is a ring homomorphism $\text{Rep}(G) \to \mathcal{C}(A, A)$; its morphisms are morphisms in $\mathcal{C}$ that are compatible with the $\text{Rep}(G)$-module structure in the obvious sense.

A module structure $\mu: \text{Rep}(G) \to \mathcal{K}(A, A)$ for $A \in \mathcal{K}K$ is equivalent to

- a natural family of $\text{Rep}(G)$-module structures in the usual sense on the groups $\mathcal{K}(D, A)$ for all $D \in \mathcal{K}K$: simply define $x : y := \mu(x) \circ y$ for $x \in \text{Rep}(G)$, $y \in \mathcal{K}(D, A)$ and notice that this recovers the homomorphism $\mu$ for $y = \text{id}_A$.

The crucial property of the universal $J_{\kappa, K}$-exact functor $F_{\kappa}$ is that it lifts the original functor $K_*(G \times \mathbb{J}): \mathcal{K}G \to \mathcal{A}_{\mathbb{C}}^{\mathbb{Z}/2}$ to a functor

$$F_{\kappa}: \mathcal{K}G \to \mathcal{M}od(\text{Rep}G)^{\mathbb{Z}/2} = \mathcal{A}_{\mathbb{C}}^{\mathbb{Z}/2}[\text{Rep}G].$$

We need a similar lifting of $G \times \mathbb{J}: \mathcal{K}G \to \mathcal{K}$ to $\mathcal{K}[\text{Rep}G]$. This requires a simple special case of exterior products in $\mathcal{KK}G$. In general, it is not so easy to
define exterior products in $KK^G$ for quantum groups because diagonal actions on $C^*$-algebras are not defined without additional structure. The only case where this is easy is if one of the factors carries the trivial coaction. This exterior product operation on the level of $C^*$-algebras also works for Kasparov cycles, that is, we get canonical maps

$$KK^G_0(A, B) \otimes KK_0(C, D) \to KK^G_0(A \otimes C, B \otimes D)$$

for all $A, B, C, D \in KK^G$. Equivalently, $(A, C) \mapsto A \otimes C$ is a bifunctor $KK^G \otimes KK \to KK^G$.

This exterior product construction yields a natural map

$$\varrho_A : \text{Rep}(G)^{\text{op}} \cong KK^G(\tau C, \tau C) \to KK^G(\tau C \otimes A, \tau C \otimes A) \cong KK^G(\tau A, \tau A)$$

for $A \in KK$, whose range commutes with the range of the map

$$\tau : KK(A, A) \to KK^G(\tau A, \tau A).$$

The ring $KK^G(\tau A, \tau A)$ acts on $KK^G(\tau A, B) \cong KK(A, G \ltimes B)$ on the right by Kasparov product. Hence so does $\text{Rep}(G)^{\text{op}}$ via $\varrho_A$. Thus $KK(A, G \ltimes B)$ becomes a left $\text{Rep}(G)$-module for all $A \in KK$, $B \in KK^G$. These module structures are natural in the variable $A$ because the images of $KK^G(\tau C, \tau C)$ and $KK(A, A)$ in $KK^G(\tau A, \tau A)$ commute. Hence they must come from a ring homomorphism

$$\mu_B : \text{Rep}(G) \to KK(G \ltimes B, G \ltimes B).$$

These ring homomorphisms are natural because the $\text{Rep}(G)$-module structures on $KK(A, G \ltimes B)$ are manifestly natural in $B$. Thus we have lifted $G \ltimes -$ to a functor

$$F : KK^G \to KK[\text{Rep} G], \quad B \mapsto (G \ltimes B, \mu_B).$$

It is clear that $\ker F = J_\kappa$. The target category $KK[\text{Rep} G]$ is neither triangulated nor Abelian. To remedy this, we use the Yoneda embedding $Y : KK \to \mathfrak{Coh}(KK)$ constructed in §2.4. This embedding is fully faithful; so is the resulting functor $KK[\text{Rep} G] \to \mathfrak{Coh}(KK)[\text{Rep} G]$.

**Theorem 73.** The functor $\mathcal{Y} \circ F : KK^G \to \mathfrak{Coh}(KK)[\text{Rep} G]$ is the universal $J_\kappa$-exact functor.

**Proof.** We omit the proof of this theorem because it is only notationally more difficult than the proof of Theorem 72. □

The category $\mathfrak{Coh}(KK)[\text{Rep} G]$ is not as terrible as it seems. We can usually stay within the more tractable subcategory $KK[\text{Rep} G]$, and many standard techniques of homological algebra like bar resolutions work in this setting. This often allows us to compute derived functors on $\mathfrak{Coh}(KK)[\text{Rep} G]$ in more classical terms.

Recall that the bar resolution of a $\text{Rep} G$-module $M$ is a natural free resolution

$$\cdots \to (\text{Rep} G)^{\otimes n} \otimes M \to (\text{Rep} G)^{\otimes n-1} \otimes M \to \cdots \to \text{Rep} G \otimes M \to M$$

with certain natural boundary maps. Defining

$$(\text{Rep} G)^{\otimes n} \otimes M = \mathbb{Z}[\hat{G}^n] \otimes M := \bigoplus_{x \in \hat{G}^n} M,$$

we can make sense of this in $\mathfrak{C}[\text{Rep} G]$ provided $\mathfrak{C}$ has countable direct sums; the $\text{Rep} G$-module structures and the boundary maps can also be defined.
If $A \in KK[\operatorname{Rep} G]$, then the bar resolution lies in $KK[\operatorname{Rep} G]$ and is a projective resolution of $A$ in the ambient Abelian category $\mathcal{C}oh(KK)[\operatorname{Rep} G]$. Hence we can use it to compute derived functors. For the extension groups, we get

$$\operatorname{Ext}^{n}_{KK^{\mathcal{C}}, \mathcal{C}}(A, B) \cong \operatorname{HH}^{n}(\operatorname{Rep}(G); KK(G \times A, G \times B));$$

here $\operatorname{HH}^{n}(R; M)$ denotes the $n$th Hochschild cohomology of a ring $R$ with coefficients in an $R$-bimodule $M$, and $KK(G \times A, G \times B)$ is a bimodule over $\operatorname{Rep} G$ via the Kasparov product on the left and right and the ring homomorphisms

$$\operatorname{Rep}(G) \to KK(G \times A, G \times A), \quad \operatorname{Rep}(G) \to KK(G \times B, G \times B).$$

Similarly, if $H : KK^{G} \to \mathfrak{A}b$ commutes with direct sums, then

$$L_{n} H(B) \cong \operatorname{HH}_{n}(\operatorname{Rep}(G); H \circ \tau(G \times B)), $$

where $H \circ \tau(G \times B)$ carries the following $\operatorname{Rep} G$-bimodule structure: the left module structure comes from $\mu_{B} : \operatorname{Rep}(G) \to KK(G \times B, G \times B)$ and the right one comes from $\mu_{A} : \operatorname{Rep}(G)^{op} \to KK^{G}(\tau A, \tau A)$ for $A = G \times B$ and the functoriality of $H$. The details are left to the reader.

5.1. The Pimsner–Voiculescu exact sequence. The reader may wonder why we have considered the ideal $\mathfrak{I}_{\kappa}$, given that the derived functors for $\mathfrak{I}_{\kappa,K}$ are so much easier to describe. This is related to the question whether $\mathfrak{I}$-equivalences are invertible. The ideal $\mathfrak{I}_{\kappa,K}$ cannot have this property because it already fails for trivial $G$. In contrast, the ideal $\mathfrak{I}_{\kappa}$ sometimes has this property. This means that the spectral sequences that we get from $\mathfrak{I}_{\kappa}$ may converge for all objects, not just for those in an appropriate bootstrap category. To illustrate this, we explain how the well-known Pimsner–Voiculescu exact sequence fits into our framework.

This exact sequence deals with actions of the group $\mathbb{Z}$; to remain in the framework of $\mathbb{R}$, we use Baaj–Skandalis duality to turn such actions into actions of the Pontrjagin dual group $T$. The representation ring of $T$ is the ring $R := \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials or, equivalently, the group ring of $\mathbb{Z}$. An $R$-bimodule in an Abelian category $\mathcal{E}$ is an object $M$ of $\mathcal{E}$ together with two commuting automorphisms $\lambda, \rho : M \to M$. The Hochschild homology and cohomology are easy to compute using the free bimodule resolution $0 \to R \otimes_{\mathbb{Z}^{2}} \to R \otimes_{\mathbb{Z}^{2}} \to R$, where $d(f) = t \cdot f \cdot t^{-1} - f$. We get

$$\operatorname{HH}_{0}(R; M) \cong \operatorname{HH}^{1}(R; M) \cong \operatorname{coker}(\lambda \rho^{-1} - 1),$$
$$\operatorname{HH}_{1}(R; M) \cong \operatorname{HH}^{0}(R; M) \cong \operatorname{ker}(\lambda \rho^{-1} - 1),$$

and $\operatorname{HH}_{n}(R; M) \cong \operatorname{HH}^{n}(R; M) \cong 0$ for $n \geq 2$. Transporting this kind of resolution to $\mathcal{E}[R]$, we get that any object of $\mathcal{E}[R]$ has an $\mathfrak{I}_{\kappa}$-projective resolution of length 1. This would fail for $\mathfrak{I}_{\kappa,K}$ because the category of $R$-modules has a non-trivial $\operatorname{Ext}^{2}$.

The crucial point is that $\mathfrak{I}_{\kappa}$-equivalences are invertible in $KK^{T}$. By Baaj–Skandalis duality, this is equivalent to the following statement: if $f \in KK^{2}(A, B)$ becomes invertible in $KK$, then it is already invertible in $KK^{\mathbb{Z}}$. We do not want to discuss here how to prove this. Taking this for granted, we can now apply Theorem 66 to all objects of $KK^{\mathbb{Z}}$.

We write down the resulting exact sequences for $KK^{\mathbb{Z}}(A, B)$ for $A, B \in KK^{\mathbb{Z}}$ because this equivalent setting is more familiar. The actions of $\mathbb{Z}$ on $A$ and $B$
provide two actions of \( \mathbb{Z} \) on \( \text{KK}_*(A, B) \). We let \( t_A, t_B : \text{KK}_*(A, B) \to \text{KK}_*(A, B) \) be the actions of the generators. Theorem 66 yields an exact sequence

\[
\ker(t_A t_B^{-1} - 1|_{\text{KK}_{n+1}(A, B)}) \to \text{KK}^Z_n(A, B) \to \ker(t_A t_B^{-1} - 1|_{\text{KK}_n(A, B)}).
\]

This is equivalent to a long exact sequence

\[
\begin{array}{ccc}
\text{KK}_1(A, B) & \longrightarrow & \text{KK}^Z_0(A, B) \\
& & \downarrow \\
\text{KK}_0(A, B) & \longrightarrow & \text{KK}^Z_0(A, B)
\end{array}
\]

Similar manipulations yield the Pimsner–Voiculescu exact sequence for the functor \( A \mapsto K_*(\mathbb{Z} \times A) \) and more general functors defined on \( \text{KK}^Z \).

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