How to go back and forth between a triangulated category $\mathcal{C}$ and some stable Abelian category $\mathcal{A}$?

Example 1: $\mathcal{C} = \text{KK}$, $\mathcal{A} = \text{Ab}_C^{\mathbb{Z}/2}$

Example 2: $\mathcal{C} = \text{KK}(X)$, $\mathcal{A} = \ ?$

Let $F : \mathcal{C} \to \mathcal{A}$ be a stable homological (e.g. $K_0 : \text{KK} \to \text{Ab}_C^{\mathbb{Z}/2}$).

For $P \in \mathcal{A}$ projective, there is $\hat{P} \in \mathcal{C}$ with $F(\hat{P}) = P$. $\mathcal{T}(\hat{P}, B) \cong \mathcal{A}(P, F(B))$.

Hence $P \mapsto \hat{P}$ is a partially defined left adjoint functor of $F$.

$P_1, P_2 \in \mathcal{A}$ projective, $\mathcal{T}(P_1, P_2) \cong \mathcal{A}(P_1, F(P_2)) \cong \mathcal{A}(P_1, P_2)$.

Now let $B \in \mathcal{C}$, let $F(B) \hookrightarrow P_0 \hookrightarrow P_1 \hookrightarrow \ldots$ be a projective resolution. Then $\mathcal{A}(P_0, F(B)) \cong \mathcal{T}(P_0, B)$ lifts the map $P_0 \to F(B)$ uniquely to $\hat{B} \to B$.

The map $\hat{P}_1 \to \hat{P}_2$ also lift uniquely to $\hat{P}_1 \to \hat{P}_2$, giving $B \hookrightarrow \hat{P}_0 \hookrightarrow \hat{P}_1 \hookrightarrow \ldots$ again a chain complex. This is a $(\text{Ker } F)$-projective resolution of $B$, and any such is of this form.

Assume that $P_n = 0$ for $n > 2$, then we may embed $\hat{P}_0 \hookrightarrow \hat{P}_1$ in an exact triangle $\hat{P}_0 \to \hat{B} \to \Sigma \hat{P}$.

$\mathcal{T}(\hat{B}, B)$ because $\mathcal{T}(-, B)$ is cohomological.

$F(\hat{P}_0) \to F(\hat{P}_1) \to F(\hat{B}) \to F(\Sigma \hat{P}) \to F(\Sigma \hat{B})$.
\[ P_i \mapsto P_0 \mapsto F(B) \mapsto \Sigma P, \rightarrow \Sigma P_0 \]

So \( Q: \hat{B} \to B \) induces isomorphism \( F(\hat{B}) \cong F(B) \)

\( \hat{B} \) is constructed out of \( (\text{Ker } F) \)-projective objects

Lemma \( T(\hat{B}, D) = 0 \) if \( F(D) = 0 \)

Proof: \( T(\hat{P}, D) = A(P, F(D)) = 0 \) use long exact seq \( T(-, D) \) for \( P_i \mapsto P_0 \mapsto \hat{B} \to \Sigma \hat{P} \)

**Thm** \( \forall D \in T \exists \) short exact sequence

\[ \text{Ext}_A^1(F(\hat{B}), \Sigma F(D)) \to T(\hat{B}, D) \to A(F(\hat{B}), F(D)) \]

Proof: Apply \( T(-, D) \) to \( P_i \mapsto P_0 \mapsto \hat{B} \to \Sigma \hat{P} \) and use \( T(\hat{P}, D) = A(P, F(D)) \)

\[ A(P, F(D)) \leftarrow A(P_0, F(D)) \leftarrow T(\hat{B}, D) \leftarrow A(\Sigma P, F(D)) \leftarrow A(\Sigma P, F(D)) \]

\[ \text{Ker } (\quad ) \leftarrow T(\hat{B}, D) \leftarrow \text{coker } (\quad ) \]
If $\varphi: \hat{B} \to B$ is an isomorphism, then the same UCT-sequence works for $B$.

Conversely, assume $\varphi$ is not an isomorphism.

Embed $\varphi$ in an exact triangle

\[ \hat{B} \to B \to C \to \Sigma \hat{B} \]  
because $F(\varphi)$ is invertible

Then $F(C) = 0$. If $B$ has UCT, then $T_{\geq 1}(B, C) = 0$ so $g = 0$.

$T_{\geq 1}(B, C) = 0$ because UCT holds for $\hat{B}$.

Long exact sequence for $T(-, C)$ implies $T_{\geq 1}(C, C) = 0$ so $\text{id}_C = 0 \Rightarrow C = 0 \Rightarrow \varphi$ is isomorphism.

So $B$ satisfies UCT iff $\varphi$ is an isomorphism.

Bootstrap class = $\exists$ things constructible from $\hat{p}, \text{PeA}$ projective

$= \exists$ things with ann($T(A, D) = 0$ if $F(D) = 0, \exists$

The UCT implies that any map $F(B) \Rightarrow F(D)$ lifts to $T(B, D)$

If $\alpha$ is invertible, then any lifting of $\alpha$ to $T(B, D)$ is also invertible. Lift $\alpha, \alpha'$ to $\tilde{\alpha}, \tilde{\alpha}'$, then $\tilde{\alpha} \cdot \tilde{\alpha}'$ and $\tilde{\alpha} \cdot \tilde{\alpha}'$ lift identity maps. So we only have to show that liftings of $\text{id}_F(\alpha)$ and $\text{id}_F(\alpha')$ are invertible. This holds because $\text{Ker} F \subseteq T(B, B)$ is nilpotent.  

$T(C, D)$
\[ \text{Ext}_A^\infty (\Sigma F(B), FC(D)) \rightarrow T(B', D) \rightarrow A(F(B), FC(D)) \]

for \( f : D \rightarrow D' \).

Assume that \( \forall B \in T \), \( F(B) \) has a length-1-projective resolution. Any \( X \in A \) with a length-1-projective resolution is \( F(B) \) for some \( B \in T \).

\( X \leq P_0 \leq P \rightarrow O \) projective resolution, lift and embed \( \overset{\sim}{P} \rightarrow \overset{\sim}{P_0} \rightarrow B \rightarrow \Sigma \overset{\sim}{P_1} \). Then \( F(B) \equiv X \)

**Example:** \( T = \text{KK}(\rightarrow \rightarrow \rightarrow) \), \( F(I \rightarrow A) = k \)-theory long exact sequence

\[
\begin{align*}
\text{k}_0(I) & \rightarrow \text{k}_0(A) \rightarrow \text{k}_0(A/I) \\
A & = \text{6-periodic chain complexes of} \\
\text{cted. Abelian groups} \\
k_1(A/I) & \leftarrow k_1(A) \leftarrow k_1(I) = R\text{-Mod} \text{ for some ring } R
\end{align*}
\]

Projective chain complexes?

\[
\begin{align*}
P_0 & \rightarrow \mathbb{Z} \rightarrow 0 \\
0 & \rightarrow \mathbb{Z} \rightarrow 0
\end{align*}
\]

\[
\text{Hom}(P_0, C_0) \cong C_0
\]

\( C_0 \) is projective in \( A \cong C_n \) projective. \( \forall n \in \mathbb{Z}/n \), \( C_0 \) is exact.
These are sums of the generators of the type
\[ Z \cong Z \to 0 \] in the above.
\[ 0 < 0 < 0 \]

\[ KK_* (\sim \sim \sim, \text{C} \cong C, I \otimes B) \cong K_* (I) \]

in general \[ KK_* (X, (x, B) \cong K_* C(B(U_x)) \text{ for } x \in X, U_x \text{ minimal open nbhd of } x, X \text{ finite} \]

\[ KK_* (\sim \sim \sim, O \otimes C, I \otimes B) \cong K_* (B) \]

\[ C = C \text{ lifts } Z \cong Z \to 0 \]
\[ 0 < 0 < 0 \]

\[ O \otimes C \text{ lifts } 0 \to Z \cong Z \]
\[ 0 < 0 < 0 \]

\[ C_0 \otimes R \cong C_0 \otimes R \text{ lifts } 0 \to 0 \to 0 \]
\[ C_0 ((0, 1)) \cong C_0 ((0, 1)) \]

\[ \overset{0}{\to} Z \cong Z \overset{0}{\to} Z \]
\[ 0 < 0 < 0 \]

\[ O \otimes C_0 \otimes I \text{ lifts } 0 \to 0 \to 0 \]
\[ 0 < 0 < 0 \]

\[ Z = Z \cong 0 \]
\[ 0 < 0 < 0 \]

\[ C_0 \otimes R \otimes C_0 ((0, 1)) \otimes C_0 ((0, 1)) \text{ lifts } Z \to 0 \to 0 \]
\[ Z < 0 < 0 \]

Direct sums of these will lift all projective objects in \( A \)
In the bootstrap class as \( \binom{C(a_1)}{A} \to \binom{C(a_1)}{A} \to \binom{C}{C} \)

\[ \mathbb{K}_* \left( \mathbb{C}_* \right) \cong \mathbb{K}_* (B_*/I) \]

How about projective resolutions? \( \mathbb{C} \leftarrow P_0 \)

Let \( C \) be a chain complex, then

- Let \( P_0 \) be the kernel of \( \mathbb{C} \leftarrow P_0 \)
- \( P_0 \) is a chain complex if free abelian groups

Long exact sequence \( H_*(C) \leftarrow H_*(P_0) \leftarrow H_*(P_1) \)

\[ e_i e_j = \delta_{ij} e_i, \quad \sum e_i = 1, \quad e_n d = e_{n-1}, \quad d^2 = 0 \]

\[ R = \bigoplus_{n \in \mathbb{Z}} e_n \mathbb{Z} \oplus \bigoplus_{n \in \mathbb{Z}} e_n d \mathbb{Z} \]

\[ F(B) \cong T(\hat{R}, B) \]

\( \text{Hom}_R(R, F(B)) \)

\[ \hat{R} = C_x \oplus C_{x_2} \oplus (C(a_1) \oplus C(a_1)) \oplus \mathbb{Z}(\ast) \]
If $X = \cdots \to A \to \cdots$ then get $A$ as above with length-1 projective resolutions.

$KK^P$