E-THEORY FOR C*-ALGEBRAS
OVER TOPOLOGICAL SPACES

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Abstract. We define E-theory for separable C*-algebras over second countable topological spaces and establish its basic properties. This includes an approximation theorem that relates the E-theory over a general space to the E-theories over finite approximations to this space. We obtain effective criteria for determining the invertibility of E-theory elements over possibly infinite-dimensional spaces. Furthermore, we prove a Universal Multicoefficient Theorem for C*-algebras over totally disconnected metrisable compact spaces.

1. Introduction

Eberhard Kirchberg [17] proved a far-reaching classification theorem for non-simple, strongly purely infinite, stable, nuclear, separable C*-algebras. Roughly speaking, two such C*-algebras are isomorphic once they have homeomorphic primitive ideal spaces – call this space $X$ – and are $\text{KK}(X)$-equivalent in a suitable bivariant K-theory for C*-algebras over $X$. To apply this classification theorem, we need tools to compute this bivariant K-theory. Following Mikael Rørdam [28] and Alexander Bonkat [3], who dealt with the simplest non-trivial case, the non-Hausdorff space with two points, Universal Coefficient Theorems for $\text{KK}(X)$ have now been established over several finite spaces $X$ in [14,22,26,27]. Here we concentrate on the special issues for infinite $X$.

Recall that Kasparov theory only satisfies excision for C*-algebra extensions with a completely positive section. Similar technical restrictions appear for all variants of Kasparov theory, including Kirchberg’s. This is a severe limitation. For instance, excision does not hold in general for extensions of the form $A(U) \rightarrow A \rightarrow A/A(U)$ for an open subset $U$, where $A(U)$ denotes the restriction of $A$ to $U$, extended by 0 to the C*-algebra over the original space, even if $A$ is nuclear. In the non-equivariant case, such technical problems are resolved by passing to E-theory, which satisfies excision for all C*-algebra extensions (see [3]). Here we define an analogue of E-theory for separable C*-algebras over a second countable topological space $X$. We establish that our new theory has the expected properties, including a universal property and exactness for all extensions of C*-algebras over $X$. If $X$ is a locally compact Hausdorff space, then our definitions agree with previous ones by Efton Park and Jody Trout in [24] and by Radu Popescu in [25]. We also formulate sufficient criteria for the natural map $E_\ast(X; A, B) \rightarrow \text{KK}_\ast(X; A, B)$ to be invertible. For instance, this works if $X$ is locally compact and Hausdorff and $A$ is a continuous field of nuclear C*-algebras over $X$.

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Our definition of $E_\ast(X; A, B)$ is based on asymptotic homomorphisms satisfying an approximate equivariance condition. An asymptotic homomorphism $\varphi_t: A \to B$, $t \in [0, \infty)$, is called approximately $X$-equivariant if for each open subset $U \subseteq X$, we have
\[
\lim_{t \to \infty} \|\varphi_t(a)||_{X \setminus U} = 0 \quad \text{for all } a \in A(U),
\]
where $\|\varphi_t(a)||_{X \setminus U}$ denotes the norm of $\varphi_t(a)$ in the quotient $B(X \setminus U) := B/B(U)$ of $B$.

Let $U = (U_n)_{n \in \mathbb{N}}$ be a countable basis for the topology of $X$. For each $n \in \mathbb{N}$, the open subsets $U_1, \ldots, U_n$ generate a finite topology $\tau_n$ on $X$. Let $X_n$ be the $T_0$-quotient of $(X, \tau_n)$, this is a finite $T_0$-space. The quotient map $X \to X_n$ allows us to view $C^\ast$-algebras over $X$ as $C^\ast$-algebras over $X_n$ for all $n \in \mathbb{N}$. Our first main result is a short exact sequence
\[
\lim_{n \to \infty} E_{n+1}(X_n; A, B) \to E_n(X; A, B) \to \lim_{n \to \infty} E_n(X_n; A, B)
\]
for all separable $C^\ast$-algebras $A$ and $B$ over $X$. This is made plausible by the observation that an asymptotic homomorphism $A \to B$ is approximately $X$-equivariant if and only if it is approximately $X_n$-equivariant for all $n \in \mathbb{N}$. Hence the space of approximately $X$-equivariant asymptotic homomorphisms is the intersection of the spaces of approximately $X_n$-equivariant asymptotic homomorphisms for $n \in \mathbb{N}$. Since there are, in general, technical problems with computing homotopy groups of intersections, we use a mapping telescope to establish the long exact sequence (1.1).

As an important application of (1.1), we give an effective criterion for invertibility of $E$-theory elements: an element in $E_n(X; A, B)$ is invertible if and only if its image in $E_n(A(U), B(U))$ is invertible for all $U \in \mathcal{O}(X)$. As a consequence, if all two-sided closed ideals of a separable nuclear $C^\ast$-algebra $A$ with Hausdorff primitive spectrum $X$ are KK-compactible, then
\[
A \otimes \mathcal{O}_\infty \otimes K \cong C_0(X) \otimes \mathcal{O}_2 \otimes K.
\]
This result solves the problem of characterising the trivial continuous fields with fibre $\mathcal{O}_2 \otimes K$ within the class of strongly purely infinite, stable, continuous fields of $C^\ast$-algebras. It is worth noting that in general the KK-compactibility of ideals does not follow from the KK-compactibility of the fibres. Indeed, there are examples of separable nuclear continuous fields $A$ over the Hilbert cube with all fibres isomorphic to $\mathcal{O}_2$ and yet such that $K_0(A) \neq 0$, see [8].

While (1.1), in principle, reduces the computation of $E_\ast(X; A, B)$ for infinite spaces $X$ to the corresponding problem for the finite approximations $X_n$, this does not yet lead to a Universal Coefficient Theorem. If $E_\ast(X_n; A, B)$ is computable by Universal Coefficient Theorems for all $n \in \mathbb{N}$, the latter will usually involve short exact sequences. Thus we have to combine two short exact sequences, as in the computation of the K-theory for crossed products by $\mathbb{Z}^2$ using the Pimsner–Voiculescu exact sequence twice. This can only be carried through if we have some extra information. In terms of the general homological machinery developed in [21], we find that the homological dimension of $E$-theory over an infinite space $X$ may be one larger than the homological dimensions of the finite approximations $X_n$. Thus it is usually at least 2, which does not suffice for classification theorems.

In fact, it is well known that filtrated K-theory cannot be a complete invariant for $C^\ast$-algebras over the one-point compactification of $\mathbb{N}$. Here we observe that the counterexample in [10] may be transported easily to any compact Hausdorff space.
The good excision properties of E-theory are particularly useful to study the E-theoretic analogue of the bootstrap class. For a finite space \( X \), the bootstrap class in \( \text{KK}(X) \) is studied in [20]. When we replace \( \text{KK}(X) \) by \( E(X) \), the technical assumptions in [20] about completely positive sections disappear, so that a \( \mathcal{C}^{*} \)-algebra \( A \) over a finite space \( X \) belongs to the E-theoretic bootstrap class if and only if all the distinguished ideals \( A(U) \) for open subsets \( U \subseteq X \) belong to the usual non-equivariant E-theoretic bootstrap class. As we shall see, the latter criterion becomes a useful \textit{definition} of the bootstrap class over an infinite space \( X \). In \( \text{KK}(X) \), this condition would not yet be sufficient for a reasonable definition of the bootstrap class.

If \( X \) is the Cantor set or, more generally, a totally disconnected metrisable compact space, then we may resolve the counterexamples mentioned above by taking into account coefficients. Our second main result is a Universal Multi-coefficient Theorem for \( E(X; A, B) \) for two \( \mathcal{C}^{*} \)-algebras \( A \) and \( B \) over \( X \). It assumes that \( A(U) \) belongs to the E-theoretic bootstrap class for all open subsets \( U \subseteq X \) and yields a natural exact sequence

\[
\text{Ext}_{C(X, A)}(K(A)[1], K(B)) \to E(X; A, B) \to \text{Hom}_{C(X, A)}(K(A), K(B)),
\]

where \( K \) denotes the K-theory of \( A \) with coefficients, viewed as a countable module over the \( \mathbb{Z}/2 \)-graded ring \( C(X, A) \) of locally constant functions from \( X \) to the \( \mathbb{Z}/2 \)-graded ring \( A \) of Bockstein operations (see [11]). As a consequence, two \( \mathcal{C}^{*} \)-algebras \( A \) and \( B \) in the E-theoretic bootstrap class over \( X \) are \( E(X) \)-equivalent if and only if \( K(A) \) and \( K(B) \) are isomorphic as \( C(X, A) \)-modules.

2. E-theory for \( \mathcal{C}^{*} \)-algebras over non-Hausdorff spaces

We recall some definitions from [20] regarding \( \mathcal{C}^{*} \)-algebras over possibly non-Hausdorff topological spaces and then introduce equivariant E-theory for them. Following the approach of Alain Connes and Nigel Higson in [5], we first describe E-theory concretely using asymptotic morphisms, then abstractly using a universal property. For a locally compact Hausdorff space \( X \), our definition is equivalent to previous ones for \( C_{0}(X) \)-algebras by Efton Park and Jody Trout in [24] and by Radu Popescu in [25].

2.1. \( \mathcal{C}^{*} \)-algebras over non-Hausdorff spaces. Here we recall some basic definitions from [20].

For a \( \mathcal{C}^{*} \)-algebra \( A \), let \( \text{Prim}(A) \) denote its primitive ideal space, equipped with the hull–kernel topology, and let \( I(A) \) be the set of ideals in \( A \), partially ordered by inclusion. For a topological space \( X \), let \( \mathcal{O}(X) \) be the set of open subsets of \( X \), partially ordered by inclusion. Both \( I(A) \) and \( \mathcal{O}(X) \) are complete lattices, that is, any subset has both an infimum and a supremum. It is shown in [13] \( \S 3.2 \) that there is a canonical lattice isomorphism

\[
(2.1) \quad \mathcal{O}((\text{Prim}(A))) \cong I(A), \quad U \mapsto \bigcap \{ p \mid p \in \text{Prim}(A) \setminus U \}.
\]

**Definition 2.2.** Let \( X \) be a topological space.

A \( \mathcal{C}^{*} \)-algebra over \( X \) is a \( \mathcal{C}^{*} \)-algebra \( A \) with a continuous map \( \psi \) from \( \text{Prim}(A) \) to \( X \).

For an open subset \( U \) of \( X \), we let \( A(U) \in I(A) \) be the ideal that corresponds to \( \psi^{-1}(U) \in \mathcal{O}(\text{Prim} A) \) under the isomorphism (2.1).
For a closed subset $S$ of $X$, we let $A(S) := A/A(X \setminus S)$. For $a \in A$, we write $\|a\|_S$ for the norm of the image of $a$ in the quotient $C^*$-algebra $A(S)$.

More generally, if $S \subseteq X$ is locally closed, that is, $S = U \setminus V$ with open subsets $V \subseteq U \subseteq X$, then we let $A(S) := A(U)/A(V)$. This quotient is independent of the choice of the open sets $U$ and $V$ with $S = U \setminus V$.

Let $A$ and $B$ be $C^*$-algebras over $X$. A $^*$-homomorphism $f : A \to B$ is called $X$-equivariant or a $^*$-homomorphism over $X$ if $f$ maps $A(U)$ into $B(U)$ for all open subsets $U$ of $X$.

Let $\mathcal{C}^*\text{alg}(X)$ be the category whose objects are the $C^*$-algebras over $X$ and whose morphisms are the $^*$-homomorphisms over $X$. Let $\mathcal{C}^*\text{sep}(X)$ be the full subcategory of separable $C^*$-algebras over $X$ with $^*$-homomorphisms over $X$ as morphisms.

We usually drop the map $\text{Prim}(A) \to X$ from our notation and simply call $A$ a $C^*$-algebra over $X$.

Although the above definition involves $X$, all that really matters is the lattice $\mathcal{O}(X)$. It is explained in [20] that it is essentially no loss of generality to assume $X$ to be sober. In that case, we may recover $X$ from the lattice $\mathcal{O}(X)$ and the map $\text{Prim}(A) \to X$ from the map $\mathcal{O}(X) \to \mathcal{I}(A)$, $U \mapsto A(U)$ (see [20] Lemma 2.25), which may be any map that commutes with finite infima and arbitrary suprema. Thus if $X$ is a second countable, sober space, a $C^*$-algebra over $X$ is a $C^*$-algebra $A$ endowed with an order preserving map $\mathcal{O}(X) \to \mathcal{I}(A)$, $U \mapsto A(U)$, which satisfies the following conditions:

1. $A(\emptyset) = 0$, $A(X) = A$,
2. $A(U_1 \cap U_2) = A(U_1) \cdot A(U_2)$,
3. $A(\bigcup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} A(U_n)$.

If a $C^*$-algebra $A$ satisfies the conditions (1) and (2) and

3'. $A(U_1 \cup U_2) = A(U_1) + A(U_2)$,

then we say that $A$ is a quasi $C^*$-algebra over $X$. If $B$ is a $C^*$-algebra over $X$ then $C_0(T, B)$ and $C_0(T, B) / C_0(T, B)$ for $T := [0, \infty)$ become quasi $C^*$-algebras over $X$, via the maps $U \mapsto C_0(T, B(U))$ and

$$U \mapsto C_0(T, B(U)) + C_0(T, B) / C_0(T, B).$$

However, they do not satisfy the condition (3) above.

Let $X$ be a locally compact Hausdorff space and let $A$ be a $C^*$-algebra over $X$. The continuous map $\text{Prim}(A) \to X$ induces a $^*$-homomorphism

$$C_b(X) \to C_b(\text{Prim}(A)) \cong ZM(A),$$

where $ZM(A)$ denotes the centre of the multiplier algebra of $A$. One verifies that $C_0(X)A$ is dense in $A$, so that $A$ becomes a $C_0(X)$-$C^*$-algebra. This yields an isomorphism of categories between $\mathcal{C}^*\text{alg}(X)$ and the category of $C_0(X)$-$C^*$-algebras with $C_0(X)$-linear $^*$-homomorphisms as morphisms by [20] Proposition 2.11.

2.2. Approximately equivariant asymptotic morphisms. Recall:

**Definition 2.3.** An asymptotic morphism between two $C^*$-algebras $A$ and $B$ is a map $\varphi : A \to C_b(T, B)$ for $T := [0, \infty)$ that induces a $^*$-homomorphism

$$\hat{\varphi} : A \to B_\infty := C_b(T, B) / C_0(T, B).$$
The map $\varphi$ is equivalent to a family of maps $\varphi_t: A \to B$ for $t \in T$ such that $t \mapsto \varphi_t(a)$ is a bounded continuous function from $T$ to $B$ for each $a \in A$. Such a family is an asymptotic morphism if and only if

$$\varphi_t(a^* + \lambda b) - \varphi_t(a)^* - \lambda \varphi_t(b) \quad \text{and} \quad \varphi_t(a \cdot b) - \varphi_t(a) \cdot \varphi_t(b)$$

converge to 0 in the norm topology for $t \to \infty$ for all $a, b \in A$, $\lambda \in \mathbb{C}$.

Two asymptotic morphisms $\varphi$ and $\varphi'$ are called equivalent if $\varphi = \varphi'$, that is, $\varphi_t(a) - \varphi'_t(a)$ converges to 0 for $t \to \infty$ for all $a \in A$.

**Definition 2.4.** An asymptotic morphism $\varphi_t: A \to B$ from $A$ to $B$ is called *approximately X-equivariant* if, for any open subset $U \subseteq X$,

$$\lim_{t \to \infty} \|\varphi_t(a)\|_{X \setminus U} = 0 \quad \text{for all } a \in A(U).$$

Let $\text{Asymp}(A, B)_X$ be the set of approximately $X$-equivariant asymptotic morphisms $A \to B$.

Our definition of $\text{Asymp}(A, B)_X$ requires $X$-equivariance only in the limit, the individual maps $\varphi_t$ need not be $X$-equivariant.

**Remark 2.6.** If $\varphi$ is equivalent to an approximately $X$-equivariant asymptotic morphism, then $\varphi$ itself is approximately $X$-equivariant.

**Lemma 2.7.** An asymptotic morphism $\varphi$ is approximately $X$-equivariant if and only if, for all closed subsets $S$ of $X$,

$$\limsup_{t \to \infty} \|\varphi_t(a)\|_S \leq \|a\|_S \quad \text{for all } a \in A.$$

**Proof.** Let $U := X \setminus S$. The limsup-criterion specialises to the definition of $X$-equivariance for $a \in A(U)$. Conversely, for any $\varepsilon > 0$ we may split $a \in A$ as $a = a_1 + a_2$ with $a_1 \in A(U)$ and $\|a_2\| < \|a\|_S + \varepsilon$ and estimate

$$\limsup \|\varphi_t(a_1)\|_S \leq \limsup \|\varphi_t(a_1)\|_S + \limsup \|\varphi_t(a_2)\|.$$

The $X$-equivariance of $\varphi$ and $a_1 \in A(U)$ imply $\lim \|\varphi_t(a_1)\|_S = 0$, and

$$\limsup \|\varphi_t(a_2)\| = \|\varphi(a_2)\| \leq \|a_2\| < \|a\|_S + \varepsilon.$$

Thus $\limsup \|\varphi_t(a)\|_S < \|a\|_S + \varepsilon$ for all $\varepsilon > 0$.

Let $U \in \mathcal{O}(X)$ and $S := X \setminus U$. The quotient map $\pi_S: B \to B(S)$ induces a map $\tilde{\pi}_S: C_b(T, B) \to C_b(T, B(S))$ whose kernel is $C_b(T, B(U))$. Condition (2.5) is equivalent to

$$\tilde{\pi}_S \circ \varphi(A(U)) \subseteq C_0(T, B(S)).$$

**Lemma 2.9.** An asymptotic morphism $\varphi$ is approximately $X$-equivariant if and only if, for all open subsets $U$ of $X$,

$$\varphi(A(U)) \subseteq C_b(T, B(U)) + C_0(T, B).$$

**Proof.** It is clear that (2.10) implies (2.8). To verify the converse, it suffices to prove

$$(\tilde{\pi}_S)^{-1}(C_b(T, B(S))) = C_b(T, B(U)) + C_0(T, B).$$

The Bartle–Graves Theorem provides a continuous section $\gamma: B(S) \to B$ of $\pi_S$. Any $f \in C_b(T, B)$ decomposes as $f = g + h$ with $g := f - \gamma \circ \tilde{\pi}_S(f)$ and $h := \gamma \circ \tilde{\pi}_S(f)$. We have $g \in C_b(T, B(U))$ and $h \in C_0(T, B)$ whenever $\tilde{\pi}_S(f) \in C_0(T, B(S))$ because $\gamma$ is continuous. \qed
For Hausdorff spaces $X$, Park and Trout [24] and Popescu [25] defined an E-theory $\mathcal{RE}_e(X;A,B)$ for $C_0(X)$-algebras based on asymptotic morphisms $\varphi$ that are \textit{asymptotically $C_0(X)$-equivariant} in the sense that $\varphi(fa) - f\varphi(a) \in C_0(T,B)$ for all $a \in A$ and $f \in C_0(X)$; equivalently, $\hat{\varphi}: A \to B_\infty$ is $C_0(X)$-linear.

**Proposition 2.11.** Let $X$ be a second countable locally compact Hausdorff space and let $A$ and $B$ be $C_0(X)$-algebras. Then an asymptotic morphism from $A$ to $B$ is \textit{asymptotically $C_0(X)$-equivariant} if and only if it is \textit{approximately $X$-equivariant}.

**Proof.** Clearly, an asymptotically $C_0(X)$-equivariant asymptotic morphism satisfies (2.10) since $A(U) = C_0(U)A$ and $C_0(U)C_b(T,B) \subseteq C_b(T,B(U))$. Conversely, let $\varphi$ be approximately $X$-equivariant. Let $B^X_\infty := C_0(X) \cdot B_\infty \subseteq B_\infty$, this is a $C_0(X)$-algebra. We are going to show that $\hat{\varphi}(C_0(U)A)$ is contained in $C_0(U) \cdot B^X_\infty = C_0(U) \cdot B_\infty$ for all $U \in \mathcal{G}(X)$. This is equivalent to the $C_0(X)$-linearity of $\varphi: A \to B^X_\infty$ by [20] Proposition 2.11.

For any $f \in C_0(U)$ and any $\varepsilon > 0$, there are a relatively compact open subset $U_\varepsilon \subseteq \overline{U} \subseteq U$ and $f_\varepsilon \in C_0(U_\varepsilon)$ with $\|f - f_\varepsilon\| < \varepsilon$. Since $A$ is a $C_0(X)$-$C^*$-algebra, the same approximation applies to all $a \in A(U) = C_0(U) \cdot A$. Therefore, it suffices to prove $\hat{\varphi}(A(U')) \subseteq C_0(U) \cdot B_\infty$ for all relatively compact open subsets $U'$ of $U$ with $\overline{U'} \subseteq U$.

Since there is a function $w$ in $C_0(U)$ with $w(x) = 1$ for all $x \in U'$, we have

$$C_0(T,B(U')) \subseteq w \cdot C_b(T,B) \subseteq C_0(U) \cdot C_b(T,B)$$

for all $n \in \mathbb{N}$. Since $\varphi$ maps $A(U')$ into $C_b(T,B(U')) + C_0(T,B)$ by (2.10), $\varphi$ maps $A(U')$ into $C_0(U) \cdot B_\infty$ for all $n \in \mathbb{N}$. 

\[\square\]

### 2.3. Homotopy of asymptotic morphisms.

**Definition 2.12.** A homotopy of asymptotic morphisms from $A$ to $B$ is an asymptotic morphism from $A$ to $C([0,1],B)$. Let $[A,B]_X$ denote the set of homotopy classes of approximately $X$-equivariant asymptotic morphisms from $A$ to $B$.

Equivalent asymptotic morphisms are homotopic.

We do not know whether there is a natural topology on $\text{Asymp}(A,B)_X$ such that $[A,B]_X = \pi_0(\text{Asymp}(A,B)_X)$. It is easy to avoid this question by using quasi-topological spaces in the sense of Edwin H. Spanier (see [30]).

**Definition 2.13.** A quasi-topological space is a set $W$ together with distinguished sets of maps $C(Y,W)$ from $Y$ to $W$ for each compact Hausdorff space $Y$, called \textit{quasi-continuous maps} $Y \to W$. These quasi-continuous maps are required to satisfy the following conditions:

- constant maps are quasi-continuous;
- a function defined on a disjoint union $Y_1 \sqcup Y_2$ is quasi-continuous if and only if its restrictions to $Y_1$ and $Y_2$ are quasi-continuous;
- if $f: Y_1 \to Y_2$ is a continuous map and $h: Y_2 \to W$ is quasi-continuous, so is $h \circ f$; and, conversely,
- if $f$ is surjective and continuous (so that $f$ is an open surjection), then $h$ is quasi-continuous provided $h \circ f$ is quasi-continuous.

Since $W$ is the set of quasi-continuous functions from the one-point space to $W$, we may also view a quasi-topological space as a contravariant functor from the
category of compact Hausdorff spaces to the category of sets with some additional properties.

We define a quasi-topology on $\text{Asymp}(A, B)_X$ by letting

$$C(Y, \text{Asymp}(A, B)_X) := \text{Asymp}(A, C(Y, B))_X$$

for each compact Hausdorff space $Y$.

Furthermore, $\text{Asymp}(A, B)_X$ has a canonical base point, the zero map. Thus $\text{Asymp}(A, B)_X$ becomes a pointed quasi-topological space.

Homotopy groups for pointed quasi-topological spaces may be defined as for ordinary topological spaces, using quasi-continuous maps instead of continuous maps. By definition, $[A, B]_X = \pi_0(\text{Asymp}(A, B)_X)$.

2.4. E-theory: Definition and universal property. The original approach of Alain Connes and Nigel Higson in [5] only works well for separable C*-algebras. The same restriction applies to our equivariant generalisation. Hence we (tacitly) assume all C*-algebras to be separable from now on. For similar reasons, we assume the underlying space $X$ to be second countable, that is, its topology must have a countable basis.

Definition 2.14. Let $X$ be a second countable topological space and let $A$ and $B$ be separable C*-algebras over $X$. Following [5], we define

$$E_0(X; A, B) := [C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K}]_X.$$ 

The orthogonal direct sum turns $E_0(X; A, B)$ into an abelian group. This also holds for $E_1(X; A, B) := E_0(X; C_0(\mathbb{R}, A), B)$.

Proposition 2.15. The composition of asymptotic morphisms induces a product

$$[A, B]_X \times [B, C]_X \to [A, C]_X.$$ 

The proof is similar to the non-equivariant case outlined in [4]. In addition to the arguments from [4, Appendix B of Chapter II], we need the following lemma to take care of approximate $X$-equivariance.

Recall that an asymptotic morphism $\varphi$ is called uniformly continuous if the map $\varphi: A \to C_b(T, B)$ is continuous. By the Bartle–Graves Theorem, every asymptotic morphism is equivalent to a uniformly continuous one.

Lemma 2.16. Let $X$ be a second countable topological space, let $A$, $B$ and $C$ be separable C*-algebras, and let $\varphi: A \to C_b(T, B)$ and $\psi: B \to C_b(T, C)$ be uniformly continuous, approximately $X$-equivariant asymptotic morphisms. Let $A_0$ be a $\sigma$-compact dense *-subalgebra of $A$. There is an increasing, continuous map $r_0: T \to T$ such that for any other increasing, continuous map $r: T \to T$ with $r(t) \geq r_0(t)$ for all $t \in T$, there is an approximately $X$-equivariant asymptotic morphism $\theta: A \to C_b(T, C)$ such that $\lim_{t \to \infty} \|\theta(t(a)) - \psi_r(t) \circ \varphi_t(a)\| = 0$ for all $a \in A_0$.

Proof. Let $(U_i)_{i=1}^\infty$ be a basis of open sets for the topology of $X$. Choose a dense sequence $(a_{ij})_{j=1}^\infty$ in $A(U_i)$ for each $i \geq 1$. We will find a map $r_0$ such that, for all $r \geq r_0$,

(i) $(\psi_r(t) \varphi_t)$ is a bounded asymptotic morphism from $A_0$ to $C$, and

(ii) $\lim_{t \to \infty} \|\psi_r(t) \circ \varphi_t(a_{ij})\|_{X \setminus U_i} = 0$ for all $i, j$. 

Then $\psi_{r(t)} \circ \varphi_t$ defines a bounded $*$-homomorphism $A_0 \to C_\infty$ by (i). It extends to a $*$-homomorphism $\hat{\theta}$ on $A$. Let $\theta: A \to C_b(T, C)$ be a lifting of $\hat{\theta}$. Then $\theta$ is approximately $X$-equivariant by (ii).

It remains to construct $r_0$. By the usual non-equivariant case, there is a continuous map $r_{00}$ such that (i) holds for all $r \geq r_{00}$. Since $\varphi(A(U_i)) \subseteq C_b(T, B(U_i)) + C_0(T, B)$, there are $f_{ij} \in C_b(T, B(U_i))$ and $g_{ij} \in C_0(T, B)$ such that $\varphi(a_{ij}) = f_{ij} + g_{ij}$ for all $i, j \geq 1$. Consider the following countable families of compact sets:

$$K_n := \bigcup_{i,j=1}^n f_{ij}[1, n + 1] \cup g_{ij}[1, n + 1] \subseteq B,$$

$$L_{i,n} := \bigcup_{j=1}^n f_{ij}[1, n + 1] \subseteq B(U_i).$$

Since $\psi$ is a uniformly continuous asymptotic morphism, we can inductively construct an increasing sequence $(s_n)_n$ such that for any $s \geq s_n$

$$\|\psi_s(x + y) - \psi_s(x) - \psi_s(y)\| < 1/n, \quad \text{for all } x, y \in K_n, \quad (2.17)$$

$$\|\psi_s(x)\| < \|x\| + 1/n, \quad \text{for all } x \in K_n. \quad (2.18)$$

Since $\psi$ is approximately $X$-equivariant and $L_{i,n} \subseteq B(U_i)$, for each $i$ there is an increasing sequence $(r_{i,n})_n$ such that

$$\|\psi_{r_{i,n}}(x)\|_{X \setminus U_i} < 1/n, \quad \text{for all } x \in L_{i,n} \text{ and all } s \geq r_{i,n}. \quad (2.19)$$

Choose an increasing continuous map $r_0: T \to T$ with $r_0(t) \geq r_{00}(t)$ and $r_0(n) \geq \max\{s_n, r_{1,n}, r_{2,n}, \ldots, r_{n,n}\}$ for all $n \geq 1$. We claim that any increasing, continuous function $r \geq r_0$ satisfies (ii). This will finish the proof.

Fix $i, j$ and $\varepsilon > 0$. Choose $n$ such that $n \geq i$, $n \geq j$ and $1/n < \varepsilon/3$. We shall show that for any $t \geq n$,

$$\|\psi_{r(t)} \circ \varphi_t(a_{ij})\|_{X \setminus U_i} < \varepsilon + \|g_{ij}(t)\|.$$

This will conclude the proof since $\lim_{t \to \infty} g_{ij}(t) = 0$ by construction. If $t \geq n$, then there is an integer $m \geq n$ such that $m \leq t < m + 1$. Therefore $f_{ij}(t)$ and $g_{ij}(t)$ are in $K_m$ and $r(t) \geq r(m) \geq s_m$. Equation (2.17) yields

$$\|\psi_{r(t)}(f_{ij}(t) + g_{ij}(t)) - \psi_{r(t)}(f_{ij}(t)) - \psi_{r(t)}(g_{ij}(t))\| < 1/m < \varepsilon/3. \quad (2.20)$$

Since $i, j \leq n \leq m$ and $t < m + 1$, we have $f_{ij}(t) \in L_{i,m}$ and $r(t) \geq r(m) \geq r_{i,m}$. Inequality (2.19) yields

$$\|\psi_{r(t)}(f_{ij}(t))\|_{X \setminus U_i} < 1/m < \varepsilon/3. \quad (2.21)$$

Similarly, (2.18) yields

$$\|\psi_{r(t)}(g_{ij}(t))\| \leq \|g_{ij}(t)\| + 1/m < \|g_{ij}(t)\| + \varepsilon/3. \quad (2.22)$$

Putting together (2.20), (2.21) and (2.22), we get

$$\|\psi_{r(t)}(a_{ij})\|_{X \setminus U_i} \leq \|\psi_{r(t)}(f_{ij}(t) + g_{ij}(t)) - \psi_{r(t)}(f_{ij}(t)) - \psi_{r(t)}(g_{ij}(t))\| + \|\psi_{r(t)}(f_{ij}(t))\|_{X \setminus U_i} + \|\psi_{r(t)}(g_{ij}(t))\| < \varepsilon + \|g_{ij}(t)\|. \quad \square$$
For any extension of separable C*-algebras \( I \hookrightarrow A \xrightarrow{p} B \), there is a canonical asymptotic morphism from \( C_0((0,1), B) \) to \( I \). If \( A \) is a C*-algebra over \( X \), then \( I \) and \( B \) become C*-algebras over \( X \) in a unique natural way, such that the given extension is an extension of C*-algebras over \( X \). Specifically, \( I(U) = I \cap A(U) \) and \( B(U) = p(A(U)) \) for all \( U \) open in \( X \).

**Proposition 2.23.** Let \( I \hookrightarrow A \rightarrow B \) be an extension of C*-algebras over \( X \). Then the associated asymptotic morphism from \( C_0((0,1), B) \) to \( I \) is approximately \( X \)-equivariant.

**Proof.** Having an extension of C*-algebras over \( X \) means that we have C*-algebra extensions

\[
I(U) \rightarrow A(U) \rightarrow B(U)
\]

for all open subsets \( U \) of \( X \). Since the map \( B(U) \rightarrow B \) is injective, this implies \( I(U) = I \cap A(U) = I \cdot A(U) \).

We fix a positive and contractive continuous approximate unit \( (u_t)_{t \in T} \) of \( I \) which is quasi-central in \( A \). The canonical asymptotic morphism

\[
\gamma: SB := C_0((0,1), B) \rightarrow C_b(T, I)
\]

is defined in two steps. First, we define a homomorphism

\[
\gamma': S A \rightarrow C_b(T, I) / C_0(T, I), \quad \gamma'(f \otimes a) := f(u_t) \cdot a.
\]

Secondly, since the restriction of \( \gamma' \) to SI is equivalent to the null asymptotic morphism, \( \gamma' \) induces an asymptotic morphism from \( SB \) to \( I \). Clearly, \( \gamma' \) is approximately \( X \)-equivariant because \( I \cdot A(U) \subseteq I(U) \). This is inherited by \( \gamma \) because \( \gamma \circ p = \gamma' \), where \( p: A \rightarrow B \) is the quotient map. \( \square \)

Let \( I \hookrightarrow B \xrightarrow{p} C \) be an extension of C*-algebras over \( X \). Let \( A \) be a C*-algebra over \( X \) and let \( \varphi: A \rightarrow C \) be an \( X \)-equivariant *-homomorphism. Let \( E \) be the C*-algebra defined by the pullback diagram

\[
\begin{array}{ccc}
0 & \rightarrow & I \\
0 \downarrow & & \downarrow \varphi \\
0 & \rightarrow & B & \xrightarrow{p} & C & \rightarrow & 0,
\end{array}
\]

that is, \( E = \{(a,b) \in A \oplus B \mid \varphi(a) = p(b)\} \). For \( U \in \mathcal{O}(X) \), set \( E(U) := E \cap (A(U) \oplus B(U)) \).

**Lemma 2.24.** \( E \) is a C*-algebra over \( X \) and \( I \hookrightarrow E \rightarrow A \) is an extension of C*-algebras over \( X \). The same conclusions hold if \( B \) and \( C \) are only quasi C*-algebras over \( X \).

**Proof.** Recall that for a quasi C*-algebra \( B \) over \( X \), the map \( U \rightarrow B(U) \) preserves only finite suprema in general. The map \( U \rightarrow E(U) \) is obviously order-preserving. The conditions \( E(\emptyset) = 0 \), \( E(\overline{X}) = E \) and \( E(U_1 \cap U_2) = E(U_1) \cap E(U_2) \) are easily verified. Let us show that \( E(U_1 \cup U_2) \subseteq E(U_1) + E(U_2) \), the reverse inclusion being obvious. Let \( (a,b) \in E(U_1 \cup U_2) \). Then \( a \in A(U_1 \cup U_2) = A(U_1) + A(U_2) \) and hence there are \( a_i \in \overline{A(U_i)} \), \( i = 1,2 \) such that \( a = a_1 + a_2 \). Since \( \varphi \) is \( X \)-equivariant, \( \varphi(a_i) \in C(U_i) \) and hence there are \( b_i \in B(U_i) \) such that \( p(b_i) = \varphi(a_i) \), \( i = 1,2 \). It follows that \( b_1 + b_2 - b \in B(U_1 \cup U_2) \) and \( p(b_1 + b_2 - b) = \varphi(a_1) + \varphi(a_2) - \varphi(a) = 0 \). Therefore, \( b_1 + b_2 - b \in I \cap B(U_1 \cup U_2) = I(U_1 \cup U_2) = I(U_1) + I(U_2) \). This shows
that there are \( x_i \in I(U_i) \), \( i = 1, 2 \), such that \( b_1 + b_2 - b = x_1 + x_2 \). It follows that 
\((a_i, b_i - x_i) \in E(U_i)\) and 
\((a, b) = (a_1, b_1 - x_1) + (a_2, b_2 - x_2)\).

It remains to show that \( E(\bigcup U_n) \) is the closure of \( \bigcup E(U_n) \) for any increasing sequence \((U_n)\) in \( \mathcal{O}(X) \). The sequence of \( C^*\)-algebras

\[
I(U) \to E(U) \to A(U)
\]

is exact for each open set \( U \). Since \( A \) and \( I \) are \( C^*\)-algebras over \( X \),

\[
A(U) = \bigcup A(U_n) = \lim A(U_n),
\]

\[
I(U) = \bigcup I(U_n) = \lim I(U_n).
\]

Since the \( C^*\)-algebra inductive limit functor is exact, we get another extension of \( C^*\)-algebras

\[
I(U) \to \bigcup E(U_n) \to A(U)
\]

because \( \lim E(U_n) = \bigcup E(U_n) \). This implies that \( E(U) \) is the supremum of \( \{ E(U_n) \} \), so that \( E \) is a \( C^*\)-algebra over \( X \).

\[\Box\]

**Theorem 2.25.** The equivariant \( E \)-theory defined above carries a composition product and hence yields a category \( \mathcal{E}(X) \). The canonical functor from the category \( C^*\text{sep}(X) \) of separable \( C^*\)-algebras over \( X \) to \( \mathcal{E}(X) \) is the universal half-exact, \( C^*\)-stable homotopy functor.

**Proof.** The composition product is described in Proposition 2.15. The same argument as in the non-equivariant case shows that it is associative. The functor \( C^*\text{sep}(X) \to \mathcal{E}(X) \) is a \( C^*\)-stable homotopy functor by definition. Next we check its exactness.

Let \( I \to E \to Q \) be an extension of \( C^*\)-algebras over \( X \). The cone

\[
C_p := \{(f, a) \in C_0((0, 1], Q) \oplus E \mid f(1) = p(a)\},
\]

\[
C_p(U) := \left(C_0((0, 1], Q(U)) \oplus E(U)\right) \cap C_p \quad \text{for } U \in \mathcal{O}(X),
\]

is a \( C^*\)-algebra over \( X \) by Lemma 2.24. The asymptotic morphism \( \gamma_t: SC_p \to SI \) induced by the extension \( SI \to CE \to C_p \) is approximately \( X \)-equivariant. There is a natural \( X \)-equivariant inclusion \( i: I \to C_p \), \( i(a) = (0, a) \). The proof of [7, Theorem 13] with no essential change yields that \( \gamma \) is a homotopy inverse of \( Si \), that is, \( [\gamma \circ Si]_X = [\text{id}_{SI}]_X \) and \( [Si \circ \gamma]_X = [\text{id}_{SC_p}]_X \). As in the non-equivariant case, this excision result and Proposition 2.15 show that \( E_0(X; A, B) := [SA \otimes K, SB \otimes K]_X \) is a periodic exact functor in both variables \( A \) and \( B \), that is, if \( I \to E \to Q \) is an extension in \( C^*\text{sep}(X) \) and \( B \) is a separable \( C^*\)-algebra over \( X \), then there are six-term exact sequences

\[
\begin{array}{ccc}
E_0(X; Q, B) & \to & E_0(X; E, B) \\
\downarrow & & \downarrow \\
E_1(X; I, B) & \leftarrow & E_1(X; E, B)
\end{array}
\]

\[
\begin{array}{ccc}
E_0(X; Q, B) & \to & E_0(X; E, B) \\
\downarrow & & \downarrow \\
E_1(X; I, B) & \leftarrow & E_1(X; E, B)
\end{array}
\]
and

\[
\begin{array}{ccc}
E_0(X; B, I) & \to & E_0(X; B, E) \\
\downarrow & & \downarrow \\
E_1(X; B, Q) & \leftarrow & E_1(X; B, I).
\end{array}
\]

The horizontal maps in both exact sequences are induced by the given maps
\(I \to E \to Q\), and the vertical maps are, up to signs, products with the class of the
approximately \(X\)-equivariant asymptotic morphism associated to the extension as in Proposition 2.23.

It remains to verify universality. Again this is similar to the proof of the non-
equivariant case in [2, Theorem 25.6.1], using Lemma 2.26 below as a substitute for
[2 Proposition 25.6.2]. \(\square\)

**Lemma 2.26.** Any element of \(E_0(X; A, B)\) may be written as \([\rho] \circ [\pi]^{-1}\) for
\(X\)-equivariant \(*\)-homomorphisms \(\rho\) and \(\pi\).

**Proof.** Let \(\varphi: A \to C_b(T, B)\) be an approximately \(X\)-equivariant asymptotic mor-
phism. We shall use Lemma 2.24 to show that the \(C^*\)-algebra
\[ E := \{(a, b) \in A \oplus C_b(T, B) : \varphi(a) - b \in C_0(T, B)\}, \]
becomes a \(C^*\)-algebra over \(X\) by
\[ E(U) := E \cap (A(U) \oplus C_b(T, B(U))). \]
As a consequence of the Bartle–Graves Theorem, for any two closed two-sided ideals
\(J_1\) and \(J_2\) in a \(C^*\)-algebra \(D\), \(C_b(T, J_1 + J_2) = C_b(T, J_1) + C_b(T, J_2)\). From this
we see that
\[ C_0(T, B) \to C_b(T, B) \to C_b(T, B)/C_0(T, B) = B_{\infty} \]
is an extension of quasi \(C^*\)-algebras over \(X\). By Lemma 2.24 its pullback under the
\(X\)-equivariant \(*\)-homomorphism \(\varphi: A \to B_{\infty}\) is an extension of \(C^*\)-algebras over \(X\):
\[ C_0(T, B) \to E \xrightarrow{\pi} A \]
with \(\pi(a, b) := \varphi(a)\). The map \(\pi\) becomes an isomorphism in \(E(X)\) because \(C_0(T, B)\)
is contractible over \(X\). Let \(\rho: E \to C_b(T, B)\) be the \(*\)-homomorphism \(\rho'(a, b) = b\).
When regarded as an asymptotic morphism from \(E\) to \(B\), \(\rho'\) is homotopic to the
constant asymptotic morphism \(\rho(a, b) = b(0)\). We have \([\varphi] \circ [\pi] = [\rho']\) because
\[ \varphi(\pi(a, b)) - \rho'(a, b) \in C_0(T, B) \text{ for all } (a, b) \in E. \]
Hence \([\varphi] = [\rho] \circ [\pi]^{-1}\). \(\square\)

2.5. **Further properties of \(E\)-theory.** Like the category \(\mathcal{M}(X)\), the category
\(\mathcal{E}(X)\) carries the additional structure of a triangulated category (see [19,23]). As
in KK-theory, the translation automorphism is the suspension functor \(A \to SA := C_0((0, 1), A)\),
and a triangle is exact if it is isomorphic to the mapping cone triangle of some \(X\)-equivariant \(*\)-homomorphism.

**Theorem 2.27.** The category \(\mathcal{E}(X)\) is triangulated.

**Proof.** The argument is essentially the same as in the appendix of [19]. The
only axiom that requires a different treatment is the one that requires each \(\varphi \in E_0(X; A, B)\) to embed in an exact triangle. Here we use the factorisation \(\varphi =
Lemma 2.26 with $X$-equivariant $*$-homomorphisms $\rho: E \to B$ and
$\pi: E \to A$. Since $[\pi]$ is invertible in $E$-theory, the mapping cone triangle

$$SB \to C_\rho \to E \xrightarrow{\rho} B$$

is isomorphic to an exact triangle

$$SB \to C_\rho \to A \xrightarrow{[\pi]} B.$$  \hfill $\square$

The proof that $E$-theory is exact shows that any extension $I \to E \to Q$ of
$C^*$-algebras over $X$ gives rise to an exact triangle $SQ \to I \to E \to Q$, where
the map $SQ \to I$ is the Connes–Higson construction (see Proposition 2.23) and
the maps $I \to E \to Q$ are the given ones. Such triangles are called extension triangles.
This works for all extensions, so that we need no admissibility assumption as in
$KK(X)$.

Since there is no admissibility hypothesis, several constructions in Kasparov
theory simplify in $E$-theory. For instance, the colimit $\lim\limits_\to (A_n, \varphi_n)$ of any inductive
system $\varphi_n: A_n \to A_{n+1}$, $n \in \mathbb{N}$, in $C^*\text{sep}(X)$ is also a homotopy colimit in $\mathcal{E}(X)$,
by the argument in [19, Section 2.4].

Proposition 2.28. If $A$ is the inductive limit of an inductive system $(A_n, \varphi_n)$ in
$C^*\text{sep}(X)$, then there is a natural short exact sequence

$$0 \to \lim\limits_\to E_1(X; A_n, B) \to E(X; A, B) \to \lim\limits_\to E(X; A_n, B) \to 0.$$  

Proof. The functor $A \mapsto E(X; A, B)$ is seen to be countably additive as in the proof
of [15, Proposition 7.1]. Then we follow the standard argument based on mapping
telescopes in [2, Section 21.3.2]. \hfill $\square$

For locally compact Hausdorff spaces, we may compare our definition of equivari-
ant $E$-theory with previous ones in [24,25]. Since we use the original Connes–Higson
model of $E$-theory instead of iterated asymptotic algebras, this does not yet follow
directly from Proposition 2.11 and [25].

Proposition 2.29. Let $X$ be Hausdorff and locally compact and let $A$ and $B$ be
$C^*$-algebras over $X$. Then $E_*(X; A, B)$ is naturally isomorphic to $RE_*(X; A, B)$.

Proof. Both theories satisfy the same universal property. Alternatively, the state-
ment follows from Proposition 2.11 and [24]. \hfill $\square$

Recall that for a compact Hausdorff space $X$, there is a canonical isomorphism

$$KK_*(X; C(X, A), B) \cong KK_*(A, B)$$

for any $C^*$-algebra $A$ and any $C^*$-algebra $B$ over $X$. The same isomorphism holds
in $E$-theory as well:

Lemma 2.30. Let $X$ be a compact Hausdorff space. Then

$$E_*(X; C(X, A), B) \cong E_*(A, B)$$

for any $C^*$-algebra $A$ and any $C^*$-algebra $B$ over $X$.

Proof. We may view $C(X, A)$ as a $C^*$-algebra over $X$ using the obvious map
Prim $C(X, A) \to X$, so that $C(X, A)(U) := C_0(U, A)$ for $U \in \mathcal{O}(X)$. We have
to show that the functor

$$\mathcal{E} \to \mathcal{E}(X), \quad A \mapsto C(X, A),$$
is left adjoint to the functor
\[ \mathcal{E}(X) \to \mathcal{E}, \quad B \mapsto B(X). \]

First of all, both maps on objects clearly induce functors on E-theory categories because of the universal properties. For the adjointness, we have to furnish the unit and counit of adjunction and verify the two zigzag equations (see [18]). The unit is the \( X \)-equivariant \(^*\)-homomorphism \( C(X,B) = C(X) \otimes B(X) \to B \) that comes from viewing a \( C^* \)-algebra \( B \) over \( X \) as a \( C(X) \)-\( C^* \)-algebra. The counit is the embedding \( A \to C(X,A)(X) = C(X,A), a \mapsto 1 \otimes a \), as constant functions. The zigzag equations are trivial to verify and hold already on the level of \(^*\)-homomorphisms. \( \square \)

**Proposition 2.31.** Let \( Y \subseteq X \) be a locally closed subset. Then there exists a natural restriction functor \( E_* (X;A,B) \to E_* (Y; r_X^* (A), r_X^* (B)) \) for \( C^* \)-algebras \( A \) and \( B \) over \( X \).

**Proof.** The restriction functor \( \mathcal{E}^* \text{sep}(X) \to \mathcal{E}^* \text{sep}(Y) \) is defined in [20] by \( r_X^* A(Z) := A(Y \cap Z) \) for \( Z \in \mathcal{O}(Y) \). It evidently maps extensions again to extensions and commutes with stabilisation. Hence it induces a functor on E-theory by the universal property. \( \square \)

### 3. Approximation by finite spaces

Let \( \mathcal{U} = (U_n)_{n \in \mathbb{N}} \) be a countable basis for the topology of \( X \). For each \( n \in \mathbb{N} \), let \( \tau_n \) be the topology generated by the open subsets \( U_1, \ldots, U_n \). That is, the subsets \( U_j \) are a subbasis for \( \tau_n \), so that the intersections
\[ U_F := \bigcap_{i \in F} U_i \]
for \( F \subseteq \{1, \ldots, n\} \) are a basis for \( \tau_n \).

Since the topology \( \tau_n \) is finite, it is pulled back from a finite \( T_0 \)-space \( X_n \); namely, we equip \( X \) with the equivalence relation
\[ x \sim_n y \iff \{1 \leq j \leq n \mid x \in U_j\} = \{1 \leq j \leq n \mid y \in U_j\} \]
for \( x, y \in X \). We may view \( \tau_n \) as a topology on the finite set \( X/\sim_n \). A point in \( X/\sim_n \) is parametrised by the set \( \{1 \leq j \leq n \mid x \in U_j\} \).

**Remark 3.1.** The minimal open neighbourhood in \( X_n \) that contains the point corresponding to \( F \subseteq \{1, \ldots, n\} \) is the image in \( X_n \) of \( U_F := \bigcap_{i \in F} U_i \).

In the following, we view \( C^* \)-algebras over \( X \) as \( C^* \)-algebras over \( (X, \tau_n) \) or, equivalently, over \( X_n := (X/\sim_n, \tau_n) \) by forgetting most of the distinguished ideals.

**Theorem 3.2.** Let \( A \) and \( B \) be \( C^* \)-algebras over \( X \), viewed as \( C^* \)-algebras over \( X_n := (X/\sim_n, \tau_n) \) for \( n \in \mathbb{N} \). Then there is a natural extension of \( \mathbb{Z}/2\)-graded abelian groups
\[ \lim_\leftarrow E_{n+1}(X_n;A,B) \to E_n(X;A,B) \to \lim_\rightarrow E_*(X_n;A,B). \]

**Proof.** Recall the description of \( \|A,B\|_X \) as the zeroth homotopy group of a quasi-topological space \( \text{Asymp}(A,B)_X \) in Section 2.3. This also applies to E-theory: we have \( E_0(X;A,B) \cong \pi_0(\Gamma_X) \) with
\[ \Gamma_X := \text{Asymp}(C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K})_X. \]
The same definitions for $X_n$ yield quasi-topological spaces $\Gamma_n := \Gamma_n X_n$ for $n \in \mathbb{N}$ with $E_0(X_n; A, B) \cong \pi_0(\Gamma_n)$. The quasi-topological spaces $\Gamma_n$ form a projective system because approximate $X_{n+1}$-equivariance implies approximate $X_n$-equivariance.

We claim that
\[ \bigcap_{n \in \mathbb{N}} \Gamma_n = \bigcap_{n \in \mathbb{N}} C(Y, \Gamma_n) \]
for each compact Hausdorff space $Y$, where $C(Y, \Gamma_n)$ denotes the space of quasi-continuous maps $Y \to \Gamma_n$.

The inclusion $C(Y, \Gamma_X) \subseteq \bigcap C(Y, \Gamma_n)$ is evident. The intersection of $C(Y, \Gamma_n)$ consists of those asymptotic morphisms that satisfy (2.10) for all $U \in \mathcal{U}$. Since the set of open subsets for which (2.10) holds is closed under arbitrary unions and $\mathcal{U}$ is a basis for the topology of $X$, this implies (2.10) for all open subsets of $X$, proving the claim.

The claim above shows that $\Gamma_X$ is the inverse limit of the projective system $\Gamma_n$.

The homotopy groups of inverse limits of ordinary topological spaces are computed by an exact sequence of the desired form if the maps $n+1 \to n$ have the homotopy covering property, see [32]. It is easy to see that this carries over to quasi-topological spaces; but in our case the maps $\Gamma_{n+1} \to \Gamma_n$ are injective and therefore cannot have the homotopy covering property. Nevertheless, we can get the desired result by following part of the argument in [32].

First we observe that [32, Theorem C], which computes the homotopy groups of homotopy equalisers remains true for quasi-topological spaces. Let $f, g: A \to B$ be two base point preserving quasi-continuous maps between pointed quasi-topological spaces. The homotopy equaliser of $f, g$ is the quasi-topological space $D(f, g)$ defined so that, for all $Y$ compact Hausdorff,
\[ C(Y, D(f, g)) = \{(a, b) \in C(Y, A) \times C(Y \times I, B) \mid f \circ a = b(1), \quad g \circ a = b(0)\}. \]

Let $Y$ be a compact Hausdorff space. Then there is an exact sequence of pointed sets
\[ \ast \to T \to [Y, D(f, g)] \to K \to \ast \]
where $[Y, X]$ denotes homotopy classes of quasi-continuous maps $Y \to X$, $K := \{a \in [Y, A] \mid f_* (a) = g_* (a)\}$, and $T$ is the orbit space for a certain canonical action of $[Y \times S^1, A]_*$ on $[Y \times S^1, B]_*$, where $[Y \times S^1, B]_*$ means that we restrict attention to quasi-continuous maps and homotopies that map $Y \times \{1\}$ to the base point.

Next we apply (3.3) to the pair of maps
\[ \text{Id}, f: \prod_{n=0}^{\infty} \Gamma_n \Rightarrow \prod_{n=0}^{\infty} \Gamma_n, \]
where $f$ is the shift map from the definition of the projective limit. Letting $\gamma_{n+1}^n: \Gamma_{n+1} \to \Gamma_n$ denote the maps in the projective system, we have
\[ f((x_n)_{n \in \mathbb{N}}) := (\gamma_{n+1}^n (x_{n+1}))_{n \in \mathbb{N}}. \]

The homotopy equaliser of $(\text{id}, f)$ is quasi-homeomorphic to the
quasi-topological space $\Gamma_\infty$ defined by

$$C(Y, \Gamma_\infty) := \{(f_n)_{n=0}^\infty \in \prod_{n \in \mathbb{N}} C([0,1] \times Y, \Gamma_n) \mid f_n(1) = \gamma_{n+1}^n(f_n(0)) \text{ for all } n \in \mathbb{N}\}.$$ 

This is a familiar mapping telescope construction. The quasi-topological version of [32, Theorem C] shows that the homotopy groups of $\Gamma_\infty$ are computed by an exact sequence of exactly the desired form.

To finish the proof of the theorem, it remains to show that the homotopy limit $\Gamma_\infty$ and the limit $\Gamma_X$ of the projective system $(\Gamma_n)$ have isomorphic $\pi_0$. Lacking the homotopy covering property used in [32], we do this by hand.

Let us describe the homotopy limit $\Gamma_\infty$ more concretely. The maps $\gamma_{n+1}^n : \Gamma_{n+1} \to \Gamma_n$ are just the inclusion maps. It is convenient to identify $C(Y, \Gamma_\infty)$ with

$$C(Y, \Gamma_\infty) = \{(f_n)_{n=0}^\infty \in \prod_{n \in \mathbb{N}} C([n, n+1] \times Y, \Gamma_n) \mid f_n(n+1) = f_{n+1}(n+1) \text{ for all } n \in \mathbb{N}\}.$$ 

We view each $f_n$ as an approximately $X_n$-equivariant asymptotic morphism from $A'$ to $C([n, n+1] \times Y, B')$, where $A' := C_0(\mathbb{R}, A) \otimes \mathbb{K}$ and $B' := C_0(\mathbb{R}, B) \otimes \mathbb{K}$. We may piece together these asymptotic morphisms to a single family of maps $\varphi_{s,t} : A' \to C(Y, B')$, $s, t \in T$, where $\varphi_{s,t}|_{s \in [n, n+1]}$ is $f_n$. That is, $\varphi_{s,t}$ is an asymptotic morphism for fixed $s$, uniformly for $s \in [n, n+1]$ for all $n$, and hence uniformly for $s$ in compact subsets of $T$; furthermore, this asymptotic morphism is (uniformly) approximately $X_n$-equivariant for $s \in [n, n+1]$ and hence for $s$ in compact subsets of $[n, \infty)$.

We map $\Gamma_X$ to $\Gamma_\infty$ by taking a constant family of asymptotic morphisms. It remains to show that this map $\Gamma_X \to \Gamma_\infty$ induces an isomorphism on homotopy classes.

Let $(\varphi_{s,t}) \in \Gamma_\infty$ and let $A_0 \subseteq A'$ be a countably generated dense subalgebra. The same considerations as in the construction of the product of asymptotic morphisms show that there is an increasing continuous function $h_0 : T \to T$ such that $\varphi_{t,h(t)} : A_0 \to B'$ extends to an $X$-equivariant asymptotic morphism for all continuous $h \geq h_0$. Here we use that an asymptotic morphism is $X$-equivariant once it satisfies (2.10) for all $U \in \mathcal{U}$. Furthermore, we may choose $h_0$ such that the convex homotopies $\varphi_{s,rh(t)+(1-r)t}$ from $\varphi_{s,t}$ to $\varphi_{s,h(t)}$ and $\varphi_{rt+(1-r)s,h(t)}$ from $\varphi_{s,h(t)}$ to $\varphi_{t,h(t)}$ are homotopies in $\Gamma_\infty$ for $h \geq h_0$. We discuss this in detail below. Thus $(\varphi_{s,t})$ is homotopic to the constant family of asymptotic morphism $(\varphi_{t,h(t)})$ in $\Gamma_\infty$, so that the map $\pi_0(\Gamma_X) \to \pi_0(\Gamma_\infty)$ is surjective. A similar argument may be applied to homotopies in $\Gamma_\infty$ and shows that two elements of $\Gamma_X(A, B)$ that become homotopic in $\Gamma_\infty$ are already homotopic in $\Gamma_X$.

Let us now show how to construct the function $h_0$ for given $(\varphi_{s,t}) \in \Gamma_\infty$. The first homotopy from $\varphi_{s,t}$ to $\varphi_{s,h(t)}$ is a homotopy of asymptotic morphisms provided $h(t) \geq t$, for obvious reasons. Thus it only remains to study the second homotopy. Let $A_0 = \{a_1, a_2, \ldots\} \subseteq A'$ be a countable dense *-subalgebra. Let $\{\lambda_1, \lambda_2, \ldots\}$ be a sequence dense in $\mathcal{C}$. Let $(U_i)_{i=1}^\infty$ be a basis of open sets for the topology of $X$. Choose a dense sequence $(a_{ij})_{j=1}^\infty$ in $A'(U_i)$ for each $i \geq 1$. 
For each integer \( m \geq 1 \) choose \( \alpha_m > 0 \) such that for all \( 1 \leq i, j, k \leq m \) and all \( t \geq \alpha_m \),

\[
\sup_{s \in [0,m+1]} \| \varphi_{s,t}(a_i^* + \lambda_k a_j) - \varphi_{s,t}(a_i)^* - \lambda_k \varphi_{s,t}(a_j) \| < 1/m, \tag{3.4}
\]

\[
\sup_{s \in [0,m+1]} \| \varphi_{s,t}(a_i a_j) - \varphi_{s,t}(a_i) \varphi_{s,t}(a_j) \| < 1/m. \tag{3.5}
\]

For each integer \( n \geq 1 \) we construct a sequence \( (\tau_{m,n})_{m=1}^{\infty} \) such that

\[
\sup_{s \in [n,m+1]} \| \varphi_{s,t}(a_{ij}) \|_{X \setminus U_i} < 1/m, \tag{3.6}
\]

for all \( 1 \leq i \leq n, 1 \leq j \leq m \) and all \( t \geq \tau_{m,n} \). Moreover, once the sequence \( (\tau_{m,n})_{m=1}^{\infty} \) is constructed, we construct the next sequence \( (\tau_{m,n+1})_{m=1}^{\infty} \) such that \( \tau_{m,n+1} \geq \tau_{m,n} \) for all \( m \geq 1 \). Let \( h_0 : T \to T \) be a continuous increasing function with \( h_0(m) \geq \max\{\alpha_m, \tau_{m,m}\} \) and \( \lim_{t \to \infty} h_0(t) = \infty \).

Let \( h \equiv h_0 \) be a continuous function. The homotopy \( \varphi_{r+t(1-r)s,h(t)} \) is defined by an element \( H \) in

\[
C([0,1] \times Y, \Gamma_{\infty}) = \left\{ (H_n)_{n=0}^\infty \in \prod_{n \in \mathbb{N}} C([0,1] \times [n,n+1] \times Y, \Gamma_n) \mid \begin{array}{l}
H_n(n+1) = H_{n+1}(n+1) \text{ for all } n \in \mathbb{N}\end{array} \right\},
\]

where for \( r \in [0,1], (H_n)_r := (\varphi_{r+t(1-r)s,h(t)})_{s \in [n,n+1], t \in T} \).

In order to verify that \( H \) is an element of \( C([0,1] \times Y, \Gamma_{\infty}) \), it is sufficient to show that for all \( i, j, k \geq 1 \)

\[
\lim_{t \to \infty} \sup_{s \in [n,n+1], r \in [0,1]} \| \varphi_{r+t(1-r)s,h(t)}(a_i^* + \lambda_k a_j)
- \varphi_{r+t(1-r)s,h(t)}(a_i)^* - \lambda_k \varphi_{r+t(1-r)s,h(t)}(a_j) \| = 0, \tag{3.7}
\]

\[
\lim_{t \to \infty} \sup_{s \in [n,n+1], r \in [0,1]} \| \varphi_{r+t(1-r)s,h(t)}(a_i a_j) - \varphi_{r+t(1-r)s,h(t)}(a_i) \varphi_{r+t(1-r)s,h(t)}(a_j) \| = 0, \tag{3.8}
\]

and that for all \( 1 \leq i \leq n, j \geq 1 \)

\[
\lim_{t \to \infty} \sup_{s \in [n,n+1], r \in [0,1]} \| \varphi_{r+t(1-r)s,h(t)}(a_{ij}) \|_{X \setminus U_i} = 0. \tag{3.9}
\]

We deal first with (3.7) and (3.8). Let \( i, j, k \geq 1 \) and \( \varepsilon > 0 \) be given. We claim that for any \( t \geq \max\{n,i,j,k,1/\varepsilon\} + 1 \), the quantities whose limits are taken in (3.7) and (3.8) are smaller than \( \varepsilon \). If \( m \) is the integer part of \( t \), then \( \max\{n,i,j,k,1/\varepsilon\} < m \leq t < m + 1 \). Moreover, for any \( s \in [n,n+1] \) and \( r \in [0,1] \), \( rt + (1-r)s \in [n,m+1] \) and \( h(t) \geq h_0(t) \geq h_0(m) \geq \alpha_m \). Since \( 1/m < \varepsilon \) our claim follows now from (3.4) and (3.5).

Let us now check (3.9). Let \( 1 \leq i \leq n, j \geq 1 \) and \( \varepsilon > 0 \) be given and suppose that \( t \geq \max\{n,j,1/\varepsilon\} + 1 \). Then there is an integer \( m \) such that \( \max\{n,j,1/\varepsilon\} < m \leq t < m + 1 \). Observe that for any \( s \in [n,n+1] \) and \( r \in [0,1] \), \( rt + (1-r)s \in [n,m+1] \) and \( h(t) \geq h_0(t) \geq h_0(m) \geq \tau_{m,m} \geq \tau_{m,n} \). Since \( 1/m < \varepsilon \), it follows from (3.6) that the quantity whose limit is taken in (3.9) is smaller than \( \varepsilon \) whenever \( t \geq \max\{n,j,1/\varepsilon\} + 1 \). \[\square\]
Theorem 3.10. Let $X$ be a second countable topological space. An element in $E_\ast(X; A, B)$ is invertible if and only if its image in $E_\ast(A(U), B(U))$ is invertible for all $U \in \mathcal{O}(X)$. 

Proof. The necessity of the condition is trivial. Next we sketch why the condition is sufficient if $X$ is a finite space. The proof is similar to the proof of a similar statement in KK-theory in [2], Proposition 4.9. If $X$ is finite, any point $x \in X$ is contained in a minimal open subset $U_x$. For a $C^\ast$-algebra $A$, let $i_x(A)$ be $A$ viewed as a $C^\ast$-algebra over $X$ concentrated at $x \in X$, that is, $i_x(A)(U) = A$ for $x \in U$ and $i_x(A)(U) = 0$ for $x \notin U$. An argument similar to the proof of [20, Proposition 3.13] yields

$$E_\ast(X; i_x(A), B) \cong E_\ast(A, B(U_x))$$

for $x \in X$, a $C^\ast$-algebra $A$ and a $C^\ast$-algebra $B$ over $X$. An argument similar to the proof of [21, Proposition 4.7] shows that objects of the form $i_x(A)$ generate $\mathcal{E}(X)$, that is, no proper triangulated subcategory of $\mathcal{E}(X)$ contains $i_x(A)$ for all $A$ (see also Proposition 4.5 below). Hence a map in $E_\ast(X; A, B)$ is invertible if the induced map $E_\ast(X; i_x(D), A) \to E_\ast(X; i_x(D), B)$ is invertible for all $x \in X$ and all $D$. By the isomorphism above, this is equivalent to invertibility of the induced map $E_\ast(D, A(U_x)) \to E_\ast(D, B(U_x))$, which is equivalent to invertibility in $E_\ast(A(U_x), B(U_x))$ for all $x$. This finishes the argument for finite $X$.

If $X$ is infinite, let $U$ be a countable basis for its topology and let $X_n$ be the resulting finite approximations to $X$. Theorem 3.2 shows that an arrow in $\mathcal{E}(X)$ is invertible if and only if its image in $\mathcal{E}(X_n)$ is invertible for all $n \in \mathbb{N}$. (The naturality of the extension in Theorem 3.2 implies that the kernel $\lim_{n\to\infty}^1 \ldots$ is nilpotent.) This reduces the general case to the finite case already settled. \qed

Theorem 3.11. Let $A$ be a separable nuclear $C^\ast$-algebra with Hausdorff primitive spectrum $X$. Suppose that each two-sided closed ideal of $A$ is KK-contractible. Then

$$A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong C_0(X) \otimes \mathcal{O}_2 \otimes \mathcal{K}.$$ 

Proof. By a result of Fell, $A$ is a continuous $C_0(X)$-algebra with nonzero simple fibres. Set $B := C_0(X) \otimes \mathcal{O}_2$. Then $0 \in E(X; A, B)$ is an $E(X)$-equivalence by Theorem 3.10. Theorem 5.4 yields $E_{\ast}(X; C, D) \cong KK_{\ast}(X; C, D)$ for $C, D \in \{A, B\}$ because $A$ and $B$ are nuclear and continuous $C_0(X)$-algebras. Hence $0 \in KK(X; A, B)$ is a $KK(X)$-equivalence, and we may apply the main result of [17] to conclude that $A \otimes \mathcal{O}_\infty \otimes \mathcal{K} \cong B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$. \qed

4. The $E$-theoretic bootstrap category

Recall that the bootstrap class $\mathcal{B}$ in $\mathcal{K}$ is the localising subcategory of the triangulated category $\mathcal{K}$ that is generated by the object $C$. Similarly, we define the $E$-theoretic bootstrap class $\mathcal{B}_E \subseteq \mathcal{E}$ as the localising subcategory of $\mathcal{E}$ generated by $C$. This is the class of all separable $C^\ast$-algebras $A$ for which $E_\ast(A, B)$ fulfills the Universal Coefficient Theorem for all $B$.

For a finite topological space $X$, a bootstrap class $\mathcal{B}(X)$ in $\mathcal{K}(X)$ is defined in [20] along similar lines. Here we follow a different approach:

Definition 4.1. Let $\mathcal{B}_E(X) \subseteq \mathcal{E}(X)$ for a second countable topological space $X$ be the class of all separable $C^\ast$-algebras $A$ over $X$ with $A(U) \in \mathcal{B}_E$ for all $U \in \mathcal{O}(X)$. 
Since the functors $\mathcal{E}(X) \to \mathcal{E}$, $A \mapsto A(U)$, are triangulated and commute with direct sums and $\mathcal{B}_E$ is a localising subcategory of $\mathcal{E}$, $\mathcal{B}_E(X)$ is a localising subcategory of $\mathcal{E}(X)$. Furthermore, if $A \in \mathcal{B}_E(X)$, then $A(Y) \in \mathcal{B}_E$ for all locally closed subsets $Y \subseteq X$ because of the extension $A(U) \to A(V) \to A(Y)$ with $Y = V \setminus U$ and suitable open subsets $U$ and $V$ in $X$.

**Proposition 4.2.** Let $X$ be a finite topological space and let $A$ be a separable C*-algebra over $X$. Then $A \in \mathcal{B}_E(X)$ if and only if $A(\overline{\{x\}}) \in \mathcal{B}_E$ for all $x \in X$.

If $A$ is tight, that is, the map $\text{Prim}(A) \to X$ is a homeomorphism, then the C*-algebras $A(\overline{\{x\}})$ for $x \in X$ are precisely the prime quotients of $A$.

**Proof.** Since $\mathcal{B}_E$ is triangulated, the class Good of locally closed subsets $Y$ of $X$ with $A(Y) \in \mathcal{B}_E$ has the following property: if $Y \subseteq Z$ and if two of $Y, Z, Z \setminus Y$ belong to Good, then so does the third. We are going to prove that a set Good of subsets must contain all locally closed subsets if it has this two-out-of-three property and contains all point closures $\overline{\{x\}}$. The proof is by induction on the length of the subspace $\overline{Y}$, that is, the length of the largest chain $x_0 < x_1 < \cdots < x_\ell$ in the specialisation preorder on the closure $\overline{Y}$. If $\ell = 0$, the subspace $Y$ is a set of closed points of $X$, and the assertion is easy.

Let $Y$ be a locally closed subset of $X$ of length $\ell$. Then $Y = \overline{Y} \setminus \partial Y$, so that it suffices to prove $\overline{Y}, \partial Y \in \text{Good}$. Therefore, we may assume without loss of generality that $Y$ is closed. Let $Z \subseteq Y$ be the set of all open points of $Y$. The difference $Y \setminus Z$ has length $\ell - 1$ and is therefore good by induction assumption. If $x \in Z$, then the closure $\overline{\{x\}}$ is good by assumption, and $\overline{\{x\}} \setminus \{x\}$ is good because its length is at most $\ell - 1$. Hence $\{x\}$ is good for all $x \in Z$. Since $Z$ is discrete, it follows that $Z$ is good. Hence so is $Y$. \hfill $\square$

Similarly, if $X$ is finite, then $A \in \mathcal{B}_E(X)$ if and only if $A(U_x) \in \mathcal{B}_E$ for all $x \in X$, where $U_x$ denotes the minimal open subset of $X$ containing $x$.

Proposition 4.2 remains true for some infinite spaces $X$ as well. For instance, let $X$ be a finite-dimensional, compact, metrisable Hausdorff space. It is proved in [8] that a continuous, separable and nuclear C(X)-algebra $A$ lies in the bootstrap class $\mathcal{B}$ if all its fibres $A(x) = A(\overline{\{x\}})$ are in $\mathcal{B}$. Applying this to all closed subsets of $X$, we get $A \in \mathcal{B}_E(X)$ under the same assumptions.

For finite spaces $X$, we may also describe the bootstrap class in terms of generators. For $x \in X$ and a C*-algebra $A$, let $i_x A$ be $A$ viewed as a C*-algebra over $X$ concentrated over $x \in X$, that is, $i_x A(U) = A$ for $x \in U$ and $i_x A(U) = 0$ for $x \notin U$. This C*-algebra over $X$ satisfies

$$\text{KK}_*(X; i_x A, B) \cong \text{KK}_*(A, B(U_x))$$

for all $B$ by [20, Proposition 3.13]. The same argument with E-theory instead of KK-theory yields

$$E_*(X; i_x A, B) \cong E_*(A, B(U_x))$$

for $x \in X$, a C*-algebra $A$ and a C*-algebra $B$ over $X$. Here $U_x$ denotes the minimal open neighbourhood of $x$, which exists because $X$ is finite. Furthermore,

$$E_*(X; A, i_x B) \cong E_*(A(\overline{\{x\}}), B)$$

as in [20], even for infinite $X$, but we will not use this in the following.
Proposition 4.5. Let $X$ be a finite topological space. Then $\mathcal{B}_E(X)$ is the localising subcategory of $\mathcal{E}(X)$ that is generated by $i_x \mathbb{C}$ for all $x \in X$. The whole category $\mathcal{E}(X)$ is generated by $C^*$-algebras of the form $i_x A$ for separable $C^*$-algebras $A$ and $x \in X$.

Proof. It is clear that $i_x \mathbb{C} \in \mathcal{B}_E(X)$ and that $\mathcal{B}_E(X)$ is localising, so that it contains the localising subcategory generated by $i_x \mathbb{C}$ for $x \in X$. The same proof as for Proposition 4.7 shows that a $C^*$-algebra $A$ over $X$ belongs to the localising subcategory of $\mathcal{E}(X)$ generated by $i_x (A(x))$ for all $x \in X$. The admissibility assumptions in [20] are only needed for KK, they become automatic in E-theory. In particular, this shows that $\mathcal{E}(X)$ is generated by $C^*$-algebras of the form $i_x A$, while $\mathcal{B}_E(X)$ is generated by $i_x A$ with $A \in \mathcal{B}_E$. Since $\mathcal{B}_E$ is generated by $\mathbb{C}$, we may replace the set of $i_x A$ with $A \in \mathcal{B}_E(X)$ by $i_x \mathbb{C}$ here. □

Theorem 4.6. Let $X$ be a second countable topological space and let $A$ and $B$ belong to $\mathcal{B}_E(X)$. An element in $E_t(X; A; B)$ is invertible if and only if it induces invertible maps $K_* (A(U)) \to K_* (B(U))$ for all $U \in \mathcal{O}(X)$.

Proof. It is well known that an element in $KK_* (A; B)$ that induces an isomorphism on $K$-theory is invertible in $KK$ provided $A$ and $B$ belong to the bootstrap category. The same argument applies to E-theory. Finally, apply Theorem 3.10 and the definition of $\mathcal{B}_E(X)$. □

5. Comparing KK- and E-theory

In the definition of E-theory, we may restrict attention to asymptotic morphisms $\varphi$ for which the maps $\varphi_t$ are all completely positive contractions. It is shown by Houghton-Larsen and Thomsen [16] that the resulting variant of E-theory agrees with Kasparov’s KK. A corresponding result for equivariant KK- and E-theory is established by Thomsen in [31]. It is a routine exercise to show that the same works in our situation.

Definition 5.1. Let $[A, B]^{cp}_X$ denote the space of homotopy classes of $X$-equivariant, completely positive, linear, contractive asymptotic morphisms $\varphi$ from $A$ to $B$, where homotopy is defined using $X$-equivariant, completely positive, linear, contractive asymptotic morphisms $A \to C_b(T, C([0, 1], B))$. $X$-equivariance means $\varphi(A(U)) \subseteq C_b(T, B(U))$ for all $U \in \mathcal{O}(X)$.

The map $\varphi: A \to C_b(T, B)$ is an $X$-equivariant, completely positive, linear contraction if and only if all the individual maps $\varphi_t: A \to B$ are $X$-equivariant, completely positive, linear contractions.

Theorem 5.2. There is a natural isomorphism

$$KK_0(X; A, B) \cong [C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K}]^{cp}_X.$$

Proof. Copy the proofs of the corresponding assertions for non-equivariant Kasparov theory and equivariant Kasparov theory for group actions in [16,31]. The main point is to go through the proof of the universal property of E-theory and to check that the variant $[C_0(\mathbb{R}, A) \otimes \mathbb{K}, C_0(\mathbb{R}, B) \otimes \mathbb{K}]^{cp}_X$ satisfies an analogous universal property, but with exactness only for extensions of $C^*$-algebras over $X$ with a completely positive contractive section over $X$. Since $\mathfrak{M}(X)$ satisfies the same universal property, the two theories must be naturally isomorphic.
Our case is somewhat closer to case of non-equivariant KK in [16] because some issues like Hilbert space representations of groups and equivariance of approximate units do not occur.

**Corollary 5.3.** Let $X$ be a second countable topological space and let $A$ be a $C^*$-algebra over $X$ which is $KK(X)$-equivalent to a $C^*$-algebra over $X$, $A'$ such that any extension $I \to E \to C_0(\mathbb{R}, A') \otimes \mathbb{K}$ of $C^*$-algebras over $X$ has an $X$-equivariant completely positive contractive linear section. Then the canonical map $KK_0(X; A, B) \to E_0(X; A, B)$ is an isomorphism for any $C^*$-algebra $B$ over $X$.

**Proof.** We may assume that $A = A'$. Any asymptotic morphism is equivalent to one with $\varphi_0 = 0$ - multiply pointwise with a suitable scalar-valued function. Hence it makes no difference whether we assume this for the definition of $[A, B]_X$ and $[A, B]_X^{\text{cp}}$. An asymptotic morphism from $A$ to $B$ with $\varphi_0 = 0$ generates an extension $C_0(T, B) \to E \to A$ with $E = \varphi(A) + C_0(T, B) \subseteq C_0(T, B)$, and two asymptotic morphisms generate the same extension if and only if they are equivalent. The asymptotic morphism itself is a section for this extension. The assumption of the corollary therefore implies $[C_0(\mathbb{R}, A) \otimes \mathbb{K}, D]_X^{\text{cp}} = [C_0(\mathbb{R}, A) \otimes \mathbb{K}, D]_X$ for all $D$. □

**Theorem 5.4.** Let $X$ be a second countable locally compact Hausdorff space, let $A$ be a nuclear and continuous $C^*$-algebra over $X$, and let $B$ be any separable $C^*$-algebra over $X$. Then the canonical map $KK_0(X; A, B) \to E_0(X; A, B)$ is an isomorphism.

**Proof.** The result follows from [24, Theorem 4.7]. Alternatively, we may argue that $A$ is $C_0(X)$-nuclear by [1, Theorem 7.2], so that it satisfies the assumptions of Corollary 5.3. □

**Theorem 5.5.** Let $X$ be a finite topological space and let $(A, \psi_A)$ and $(B, \psi_B)$ be $C^*$-algebras over $X$. The canonical map

$$KK_*(X; A, B) \to E_*(X; A, B)$$

is an isomorphism if $A$ belongs to the bootstrap class in $\mathcal{R}(X)$ defined in [20]. In particular, this applies if the $C^*$-algebra $A(X)$ is nuclear.

**Proof.** If $A$ belongs to the bootstrap class of [20], then we may compute $KK_*(X; A, B)$ by a spectral sequence whose first page only involves the K-theory groups of $A(U)$ and $B(U)$ for minimal open subsets $U$ in $X$. The arguments in [20] only use the universal property of $\mathcal{R}(X)$ and work equally well for $\mathcal{E}(X)$, with some simplifications because we do not have to worry about equivariant completely positive sections of various extensions. Thus there is an analogous spectral sequence computing $E_*(X; A, B)$, and it has the same first page as the spectral sequence computing $KK_*(X; A, B)$. The canonical map $\mathcal{R}(X) \to \mathcal{E}(X)$ provides a morphism between these spectral sequences, which is an isomorphism on the first page and thus on all later pages. Hence the two spectral sequences are isomorphic, so that $KK_*(X; A, B) \cong E_*(X; A, B)$. □

**Example 5.6.** We exhibit an extension of nuclear $C^*$-algebras over $[0, 1]$ which is not excisive for $KK([0, 1]; \omega B)$. Consider the extension of $C^*$-algebras over $[0, 1]$ $0 \to C_0[0, 1] \to C[0, 1] \xrightarrow{\pi} \mathbb{C} \to 0$, where $\pi(f) = f(1)$. We claim that the mapping cone $C_\pi$ is not $KK([0, 1])$-equivalent to $\ker(\pi) = C_0[0, 1]$ and that $KK([0, 1]; SC, C_0[0, 1]) \neq E([0, 1]; SC, C_0[0, 1])$. 

Here $SC$ is regarded as a $C[0,1]$-algebra via the multiplication $f \cdot g = f(1)g$ for $f \in C[0,1]$ and $g \in SC$. Let us address first the second part of the claim. It is convenient to work with asymptotic morphisms parametrised by $t \in [0,1)$. For each such $t$ consider the map $\nu_t : [0,1] \to [0,1]$,

$$\nu_t(s) = \begin{cases} 0 & \text{if } 0 \leq s < t, \\ \frac{s-t}{1-t} & \text{if } t \leq s \leq 1. \end{cases}$$

Define a continuous family of $^*$-homomorphisms $\varphi_t : SC \to C_0[0,1], t \in [0,1)$ by $\varphi_t(\exp(2\pi is) - 1) := \exp(2\pi i\nu_t(s)) - 1$. It is easily verified that the asymptotic homomorphism $(\varphi_t)$ is asymptotically $[0,1]$-equivariant since $\exp(2\pi i\nu_t(s)) - 1$ is supported on $[t,1)$. Set $A = SC$ and $B = C_0[0,1]$. We observe that the class of $(\varphi_t)$ in $E([0,1];A,B)$ is non-zero since its image in $\text{Hom}(K_1(A(0,1)),K_1(B(0,1))) \cong \text{Hom}(Z,Z)$ is equal to $\text{id}_Z$. On the other hand,

$$KK_*([0,1]; A,B) = KK_* (SC, \bigcap_n B((1 - 1/n,1])) = KK_* (SC, \{0\}) = 0,$$


We verify the first part of the claim. The Puppe sequence for $KK([0,1]; A,B)$ associated to the map $\pi$ yields $KK([0,1],[C_\pi, B]) = 0$ since $KK_*([0,1],[C[0,1], B]) = KK_* (C, B) = 0$ and $KK_* ([0,1]; C,B) = 0$ as argued above. At the same time, $KK_* ([0,1]; B, B) \neq 0$ since the natural map

$$KK_*([0,1]; B, B) \to \text{Hom}(K_1(B(0,1)),K_1(B(0,1)) \cong Z$$

sends $[\text{id}_B]$ to $1$.

6. A universal coefficient theorem for $C^*$-algebras over totally disconnected spaces

In this section, we study $C^*$-algebras over a totally disconnected compact metrisable space $X$. Our goal is to construct a Universal Coefficient Theorem that computes $E_*(X; A,B)$ for $A, B \in \mathcal{B}_E(X)$. For this purpose, we use filtrated $K$-theory with coefficients and obtain a Universal Coefficient exact sequence that generalises the Multicoefficient Theorem of [11]. In order to explain the key role of filtrated $K$-theory with coefficients, we also revisit an example from [10] showing that the spectral sequence generated by filtrated $K$-theory does not degenerate to an exact sequence.

In this section, all $C^*$-algebras are assumed separable and all groups countable. Let $\mathcal{P} \subseteq \mathbb{N}$ be the set consisting of $0$ and all prime powers. The relevance of the set $\mathcal{P}$ in the Universal Multicoefficient Theorem is that the groups $\mathbb{Z}/p$ for $p \in \mathcal{P}$ are exactly the indecomposable abelian groups.

For $p \in \mathcal{P}$ let $I_p$ be the mapping cone of the unital $^*$-homomorphism $\mathbb{C} \to M_p(\mathbb{C})$. For $p = 0$, we let $I_0 := \mathbb{C}$. It is convenient to denote $I_p$ by $I^0_p$ and its suspension $SI_p$ by $I^1_p$. Then for a $C^*$-algebra $A$:

$$K_1(A; \mathbb{Z}/p) := KK_1(I_p, A) \cong KK(I^i_p, A), \quad i = 0,1.$$

Let us set $I := \bigoplus_{p \in \mathcal{P}} I_p$ and consider the ring $KK_* (I, I)$ with multiplication given by the Kasparov product. The non-unital subring

$$\Lambda = \bigoplus_{p,q \in \mathcal{P}} KK_* (I_p, I_q)$$
of KK\(_k(\mathbb{I}, \mathbb{I})\) is called the ring of B\"{o}ckstein operations. It consists of matrices indexed by \(\mathcal{P} \times \mathcal{P}\) with only finitely many non-zero entries \(\lambda_{pq} \in KK\(_k(\mathbb{I}_p, \mathbb{I}_q).\) The Kasparov product

\[ KK\(_k(\mathbb{I}_p, \mathbb{I}_q) \times KK\(_k(\mathbb{I}_q, A) \to KK\(_k(\mathbb{I}_p, A) \]

induces a natural \(\Lambda\)-module structure on the \(\mathbb{Z}/2 \times \mathcal{P}\)-graded group

\[ \mathbb{K}(A) = \bigoplus_{p \in \mathcal{P}} KK\(_k(A; \mathbb{Z}/p). \]

The KK-class \(x^i_p\) of id\(_{\mathbb{I}_p}\) generates the group \(KK\(_k(\mathbb{I}_p, \mathbb{I}_p) \cong KK\(_k(\mathbb{I}_p, \mathbb{I}_p).\) We shall work with \(\mathbb{Z}/2 \times \mathcal{P}\)-graded \(\Lambda\)-modules \(M = (M^i_p)\) such that for \(\lambda \in KK\(_k(\mathbb{I}_q, \mathbb{I}_k)\) and \(m \in M^i_p, \lambda m \in M^{i+\ell} \) if \(k = p\) and \(\lambda m = 0\) if \(k \neq p.\) We also ask that \(x^i_p\) acts as the identity automorphism on \(M^i_p.\) In particular, this implies that \(pM^i_p = 0.\) These assumptions are modelled on the case \(M = \mathbb{K}(A)\) where \(M^i_p = KK\(_k(\mathbb{I}_p, A)\).

**Definition 6.1.** A \(\Lambda\)-module isomorphic to \(\mathbb{K}(\mathbb{I}_p^i)\) for some \((i, p) \in \mathbb{Z}/2 \times \mathcal{P}\) is called basic.

**Lemma 6.2.** For all \((i, p) \in \mathbb{Z}/2 \times \mathcal{P}, \mathbb{K}(\mathbb{I}_p^i) = \Lambda \cdot x^i_p.\) The basic \(\Lambda\)-modules are projective in the category of \(\mathbb{Z}/2 \times \mathcal{P}\)-graded \(\Lambda\)-modules.

**Proof.** The first part follows because \(KK\(_k(\mathbb{I}_p, \mathbb{I}_p) \cong KK\(_k(\mathbb{I}_p, \mathbb{I}_p)\) and \(x^i_p = [id_{\mathbb{I}_p}]\) is idempotent. For the second part we observe that if \(\lambda x^i_p = 0\) for some \(\lambda \in KK\(_k(\mathbb{I}_q, \mathbb{I}_k)\) then either \(k \neq p\) or \(\lambda = 0.\) This shows that if \(\pi: B \to C\) is a surjective morphism of \(\Lambda\)-modules, then any morphism \(\varphi: \Lambda x^i_p \to C\) lifts to a morphism \(\Phi: \Lambda x^i_p \to B\) defined by \(\Phi(\lambda x^i_p) = \lambda b^i_p, \lambda \in \Lambda,\) where \(b^i_p\) is some lifting of \(\varphi(x^i_p).\) \(\square\)

We give a very concise proof of the following result from [11].

**Proposition 6.3.** Let \(A\) and \(B\) be separable \(C^*\)-algebras and suppose that \(A\) is in the bootstrap class \(\mathcal{B}\) with \(KK\(_k(A)\) finitely generated. Then \(KK(A, B) \cong Hom\(_\Lambda(\mathbb{K}(A), \mathbb{K}(B)).\)

**Proof.** Both sides are additive in the first variable. Thus by the UCT we may assume that \(A = \mathbb{I}_p^i\) for some \((i, p) \in \mathbb{Z}/2 \times \mathcal{P}.\) Let us observe that any element \(h \in Hom\(_\Lambda(A x^i_p, \mathbb{K}(B)\))\) is completely determined by \(h(x^i_p) \in KK\(_k(\mathbb{I}_p, \mathbb{I}_p).\) Moreover, the image of \(h(x^i_p)\) under the map \(KK(\mathbb{I}_p^i, B) \to Hom\(_\Lambda(\mathbb{K}(\mathbb{I}_p^i), \mathbb{K}(B).\))\) is precisely \(h.\) Indeed, the Kasparov product \(KK(\mathbb{I}_p^i, \mathbb{I}_p^i) \times KK(\mathbb{I}_p^i, B) \to KK(\mathbb{I}_p^i, B)\) gives \([id_{\mathbb{I}_p^i}] \times \alpha = \alpha.\) \(\square\)

If \(A\) is a separable \(C^*\)-algebra over a zero-dimensional space \(X,\) then \(\mathbb{K}(A)\) has a natural structure of module over the ring \(C(X, \Lambda)\) of locally constant functions from \(X\) to \(\Lambda.\) This is easily seen by observing that \(A \cong \bigoplus_{k=1}^n A(U_k)\) for any clopen partition \((U_k)_{k=1}^n\) of \(X.\) A \(C^*\)-algebra over \(X\) is called elementary if it is isomorphic to \(\bigoplus_{k=1}^n C(U_k, A_k),\) where \((U_k)_{k=1}^n\) is a clopen partition of \(X,\) each \(A_k\) is a separable \(C^*\)-algebra in the bootstrap class, and \(KK(A_k)\) is finitely generated. If \(A\) is elementary, then the \(C(X, \Lambda)\)-module \(\mathbb{K}(A)\) is isomorphic to \(\bigoplus_{k=1}^n C(U_k, \mathbb{K}(A_k)).\) Since \(KK\(_k(A_k)\) is finitely generated, it follows from the UCT that \(A_k\) is KK-equivalent to a finite direct sum of \(\mathbb{I}_p^i\)s, so that \(\mathbb{K}(A_k)\) is \(\Lambda\)-projective by Lemma 6.2. It follows easily that the \(C(X, \Lambda)\)-module \(\mathbb{K}(A_k)\) is projective.
Lemma 6.4. Suppose that $M$ is isomorphic to the inductive limit of an inductive system $(M_j)$ of projective $C(X, \Lambda)$-modules. Then for any $C(X, \Lambda)$-module $N$ there is a natural isomorphism

$$\lim_{\leftarrow}^1 \text{Hom}_{C(X, \Lambda)}(M_j, N) \cong \text{Ext}_{C(X, \Lambda)}(M, N).$$

Proof. Set $R = C(X, \Lambda)$. The extension

$$0 \to \bigoplus_{j \in \mathbb{N}} M_j \xrightarrow{\text{Id} - S} \bigoplus_{j \in \mathbb{N}} M_j \to M \to 0,$$

where $S$ is the natural shift map, is a projective resolution of $M$. Since $\bigoplus_{j \in \mathbb{N}} M_j$ is projective, we have an exact sequence

$$\text{Hom}_R \left( \bigoplus_{j \in \mathbb{N}} M_j, N \right) \xrightarrow{(\text{Id} - S)^*} \text{Hom}_R \left( \bigoplus_{j \in \mathbb{N}} M_j, N \right) \to \text{Ext} R(M, N) \to 0,$$

where the first map identifies with the first map of the exact sequence

$$\prod_{j \in \mathbb{N}} \text{Hom}_R(M_j, N) \to \prod_{j \in \mathbb{N}} \text{Hom}_R(M_j, N) \to \lim_{\leftarrow}^1 \text{Hom}_R(M_j, N) \to 0$$

that defines $\lim_{\leftarrow}^1$. Thus the two maps have isomorphic cokernels. \(\square\)

Proposition 6.5. Let $A$ be a separable nuclear continuous $C^*$-algebra over a totally disconnected compact metrisable space $X$. Suppose that each fibre of $A$ belongs to the bootstrap class $B$. Then $A$ is $\text{K(K)}(X)$-equivalent to the inductive limit of an inductive system of elementary $C(X)$-algebras.

Proof. [8, Theorem 2.5] shows that $A$ is $\text{K(K)}(X)$-equivalent to a unital continuous $C(X)$-algebra $A^d$ whose fibres are Kirchberg algebras. Thus we may assume that $A = A^d$. By [12, Theorem 3.6], there is a sequence $(A_n)_{n=1}^{\infty}$ of elementary unital $C(X)$-subalgebras of $A$ which is exhausting $A$ in the sense that for every finite subset $F$ of $A$, $\lim_{n \to \infty} \text{dist}(F, A_n) = 0$. Since $A_n$ is locally trivial and its fibres are weakly semiprojective ( [2, Section 3]) each inclusion map $\gamma_n : A_n \hookrightarrow A$ can be perturbed to some $C(X)$-linear unital $*$-monomorphism $\gamma_{n,n+k} : A_n \to A_{n+k}$ with $\|\gamma_n(a) - \gamma_{n,n+k}(a)\| < 1/2^n$ for $a$ in a prescribed finite subset of $A_n$. It follows that after passing to a subsequence of $(A_n)$ we can represent $A$ as the inductive limit of a system $(A_{nk}, \gamma_{nk,nk+1})$ of elementary $C(X)$-algebras. \(\square\)

Lemma 6.6. Let $A$ and $B$ be separable $C(X)$-algebras over a totally disconnected compact metrisable space $X$ and suppose that $A$ is elementary. Then $\text{K(K)}(X; A, B) \cong \text{Hom}_{C(X, \Lambda)}(K(A), K(B))$.

Proof. Write $A = \bigoplus_{i=1}^k C(U_i, D_i)$ where $U_1, \ldots, U_k$ is a clopen partition of $X$ and each $D_i$ is in the bootstrap class with $K_*(D_i)$ finitely generated. We have $\text{K(K)}(X; A, B) \cong \bigoplus_{i=1}^k \text{K(K)}(U_i; A(U_i), B(U_i))$ and

$$\text{Hom}_{C(X, \Lambda)}(K(A), K(B)) \cong \bigoplus_{i=1}^k \text{Hom}_{C(U_i, \Lambda)}(K(A(U_i)), K(B(U_i))).$$
Thus we may assume that $A = C(X, D)$. In this case, the assertion follows from the commutative diagram

$$
\begin{array}{ccc}
\text{KK}(X; C(X, D), B) & \longrightarrow & \text{Hom}_{C(X, A)}(\text{KK}(C(X, D)), \text{KK}(B)) \\
\cong & & \cong \\
\text{KK}(D, B) & \longrightarrow & \text{Hom}_A(\text{KK}(D), \text{KK}(B))
\end{array}
$$

The bottom horizontal map of the diagram is bijective by Proposition 6.3, the left vertical may by Lemma 2.30 The right vertical map is bijective because

$$\text{KK}(C(X, D)) \cong C(X, \text{KK}(D)) \cong C(X, Z) \otimes \text{KK}(D) \cong C(X, Z) \otimes_A \text{KK}(D) \cong C(X, A) \otimes_A \text{KK}(D)$$

and

$$\text{Hom}_{C(X, A)}(C(X, A) \otimes_A \text{KK}(D), \text{KK}(B)) \cong \text{Hom}_A(\text{KK}(D), \text{KK}(B)). \quad \Box$$

Lemma 6.7. Any separable $C(X)$-algebra over a totally disconnected compact metrisable space $X$ is isomorphic to the inductive limit of a sequence of locally trivial separable $C(X)$-algebras.

Proof. Let $A$ be a separable $C(X)$-algebra over $X$. If $U$ is a finite clopen cover of $X$ we denote by $A_U$ the locally trivial continuous $C(X)$-algebra $\bigoplus_{U \in U} C(U) \otimes A(U)$. For each $x \in U$ the fibre $A_U(x)$ is $A(U)$. There is a natural morphism of $C(X)$-algebras $\alpha_U: A_U \to A$ which maps $(f_U \otimes a_U)_{U \in U}$ to $\sum_{U \in U} f_U a_U$.

If $V$ is a closed subset of $U$ we have a natural restriction homomorphism $C(U) \otimes A(U) \to C(V) \otimes A(V)$, which maps $f \otimes a$ to $f|_V \otimes \pi_V(a)$. Therefore, if $V$ is a finite clopen cover of $X$ which refines $U$, there is a natural morphism of $C(X)$-algebras $\alpha^V_U: A_U \to A_V$ such that $\alpha^V_V = \alpha_U$.

Let $(U_n)_n$ be an infinite sequence of finite clopen covers of $X$, with $U_{n+1}$ refining $U_n$, and such that diam$(U_n) \to 0$ with respect to some metric inducing the topology of $X$. Set $A_n = A_{U_n}, \alpha_n = \alpha_{U_n}$ and $\alpha_n^m = \alpha_{U_n}^m$. We claim that the natural morphism $\text{lim}(A_n, \alpha_n^m) \to A$ is an isomorphism. This morphism is surjective since each $\alpha_n$ is surjective. To prove its injectivity, it suffices to show that if $F \in A_n$ satisfies $\alpha_n(F) = 0$, then for any $\varepsilon > 0$ there is $m > n$ such that $\|\alpha_n^m(F)\| \leq \varepsilon$. By localising at each element of $U_n$, we may assume that $A_n = C(X) \otimes A(X)$ and regard $F$ as a continuous function $F: X \to A(X)$. Since $F$ is continuous, each $x \in X$ has a neighbourhood $V_x$ such that $\|F(x) - F(y)\| < \varepsilon/2$ for all $y \in V_x$. Since $A(X)$ is a $C(X)$-algebra, for each $a \in A(X)$, the map $x \mapsto \|\pi_x(a)\|$ is upper semi-continuous. The assumption $\alpha_n(F) = 0$ implies that $\pi_x(F(x)) = 0$ for all $x \in X$. Thus, after shrinking each $V_x$ if necessary, we may arrange that $\|\pi_x(F(x))\| < \varepsilon/2$ for all $z \in V_x$. It follows that for any $y, z \in V_x$,

$$\|\pi_z(F(y))\| \leq \|\pi_z(F(y) - F(x))\| + \|\pi_z(F(x))\| < \varepsilon.$$

Extract now a finite cover $V_{x_1}, \ldots, V_{x_r}$ of $X$. Since diam$(U_m) \to 0$ there is $m > n$ such that each element of $U_m$ is contained in some $V_{x_i}$. It follows that $\|\alpha_m^m(F)\| \leq \varepsilon$. □

Proposition 6.8. Any separable $C(X)$-algebra over a totally disconnected compact metrisable space $X$ is $E(X)$-equivalent to a continuous separable $C(X)$-algebra.
Proof. For a given C(X)-algebra A, let \((A_n, \alpha_n^m)\) be the corresponding inductive system constructed as in the proof of the previous lemma. Let

\[ T(A_n, \alpha_n^m) = \left\{ (f_n) \in \bigoplus_{n \in \mathbb{N}} C([n, n+1], A_n) \mid f_{n+1}(n+1) = \alpha_n^{n+1}(f_n(n+1)) \right\} \]

be the associated mapping telescope. Since the mapping telescope construction is functorial, there is a natural C(X)-linear *-homomorphism \(\alpha: T(A_n, \alpha_n^m) \rightarrow T(A, \text{id}_A) \cong S_A\).

Arguing as in the paragraphs following the proof of \([19]\) Proposition 2.6, it follows that \(\alpha\) is an E(X)-equivalence. Indeed, let \(T(A_m, \alpha_m^m)\) be the variant of \(T(A_m, \alpha_m^m)\) where we require \(\lim_{t \to \infty} \alpha_m(f_m(t))\) to exist in \(A\) instead of \(\lim f_m(t) = 0\). The algebra \(T(A_m, \alpha_m^m)\) is contractible over \(X\) in a natural way. There is a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & T(A_m, \alpha_m^m) \\
\alpha & \downarrow & \downarrow \\
0 & \rightarrow & T(A, \text{id})
\end{array}
\]

whose rows are short exact sequences. Since the algebras in the middle are contractible, it follows that \(\alpha\) induces an E(X)-equivalence. We conclude by observing that \(T(A_n, \alpha_n^m)\) is a continuous C(X)-algebra since it is a C(X)-subalgebra of a direct sum of continuous C(X)-algebras.

\[ \square \]

Proposition 6.9. A separable and nuclear C(X)-algebra \(A\) over a totally disconnected compact metrisable space \(X\) belongs to the bootstrap class \(B_{E}(X)\) if and only if all its fibres are in the bootstrap class \(B_{E}\).

Proof. By Propositions \(6.8\) and \(2.29\), we may assume that \(A\) is a continuous C(X)-algebra. By a result of \([8]\), a separable nuclear continuous C(X)-algebra over a finite-dimensional compact metrisable space \(X\) belongs to \(B\) if and only if all its fibres belong to \(B\). This concludes the proof, since a nuclear C*-algebra belongs to \(B\) if and only if it belongs to \(B_{E}\).

\[ \square \]

Proposition 6.10. Let \(A\) be a separable C(X)-algebra over a totally disconnected compact metrisable space \(X\). If \(A(U)\) is E-equivalent to a separable nuclear C*-algebra for each clopen set \(U \subset X\), then \(A\) is E(X)-equivalent to a separable, continuous, nuclear C(X)-algebra.

Proof. The proposition applies for instance when \(A\) belongs to the bootstrap class \(B_{E}(X)\). It was shown in \([8]\) Lemma 2.2] that \(A\) is KK(X)-equivalent to a C(X)-algebra \(A'\) such that \(A' \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong A'\) and that \(A'\) contains a full projection. Thus we may assume that \(A\) itself has these properties. Let \((A_n, \alpha_n^m)\) be the inductive system constructed in the proof of Lemma \(6.7\), that is, \(A_n\) is of the form \(\bigoplus_{k=1}^{\pi[n]} C(U_k) \otimes A(U_k)\) with a partition into clopen sets \(U_k\). It is clear that \(A(U_k) \cong A(U_k) \otimes \mathcal{O}_\infty \otimes \mathbb{K}\) and that \(A(U_k)\) contains a full projections. By assumption, each C*-algebra \(A(U_k)\) is E-equivalent to some nuclear separable C*-algebra and hence it is E-equivalent to some stable Kirchberg algebra \(D_k\). For each \(k\), Kirchberg’s Classification Theorem \([29\) Theorem 8.3.3] yields a *-homomorphism \(\eta_k: D_k \rightarrow A(U_k)\) which lifts the given E-equivalence. Moreover, we may arrange that \(\eta_k\) decomposes as \(\eta_k = \mu_k \oplus \theta_k\), where \(\theta_k\) is a full *-monomorphism that factors
through the stable Cuntz algebra $O_2 \otimes K$. Extending the $\eta_k$ by $C(X)$-linearity and taking their direct sum, we get a $C(X)$-linear monomorphism $\varphi_n : B_n \to A_n$, where $B_n := \bigoplus_{k=1}^{\infty} C(U_k) \otimes D_k$. Moreover, each $\varphi_n$ induces an equivalence in $\mathcal{E}(X)$. Another application of [29, Theorem 8.3.3] yields $C(X)$-linear $*$-monomorphisms $\beta_n+1 : B_n \to B_{n+1}$ such that for each $n$ the diagram

$$
\begin{array}{ccc}
A_n & \xrightarrow{\alpha_n+1} & A_{n+1} \\
\varphi_n \uparrow & & \uparrow \varphi_{n+1} \\
B_n & \xrightarrow{\beta_n+1} & B_{n+1}
\end{array}
$$

commutes in $\mathcal{E}(X)$ and hence in the category $\mathcal{R}(X)$ (since each $D_k$ is nuclear). The uniqueness part of [29, Theorem 8.3.3] shows that we may arrange that the diagram above commutes up to unitary homotopy. By [9, Section 2] this gives a $C(X)$-linear $*$-homomorphism $\varphi : B \to C_0(T, A)/C_0(T, A)$, where $B$ is the limit of the inductive system $(B_n, \beta_n+1)$, such that the diagram

$$
\begin{array}{ccc}
A_n & \xrightarrow{} & A \\
\varphi_n \uparrow & & \uparrow \varphi \\
B_n & \xrightarrow{} & B
\end{array}
$$

commutes in $\mathcal{E}(X)$. By Proposition 2.28 for any separable $C(X)$-algebra $D$ there is a commutative diagram with exact rows

$$
\begin{array}{ccc}
\lim^1 \text{E}_1(X; A_i, D) & \xrightarrow{} & \text{E}(X; A, D) \xrightarrow{} \lim \text{E}(X; A_i, D) \\
\varphi_n^* \downarrow & & \varphi^* \downarrow \varphi_n^* \\
\lim^1 \text{E}_1(X; B_i, D) & \xrightarrow{} & \text{E}(X; B, D) \xrightarrow{} \lim \text{E}(X; B_i, D).
\end{array}
$$

Since the maps $\varphi_n^*$ are bijective by construction, we conclude that $A$ is $E(X)$-equivalent to the nuclear continuous $C(X)$-algebra $B$. \qed

**Theorem 6.11.** Let $A$ and $B$ be separable $C(X)$-algebras over a totally disconnected compact metrisable space $X$. If $A$ is in the bootstrap class $B_E(X)$, then there is an exact sequence

$$\text{Ext}_{C(X; A)}(K(A), K(SB)) \to \text{E}(X; A, B) \to \text{Hom}_{C(X; A)}(K(A), K(B)).$$

**Proof.** By Proposition 6.10 we may assume that $A$ is a continuous nuclear $C(X)$-algebra with all its fibres in the bootstrap class $B$. Then $E(X; A, B) \cong \text{KK}(X; A, B)$ by Theorem 5.4. By Proposition 6.5 we may also assume $A \cong \varinjlim A_n$ for an increasing sequence $(A_n)_{n=1}^{\infty}$ of elementary $C^*$-subalgebras of $A$. Then we can apply the $\lim^1$-sequence for nuclear continuous $C(X)$-algebras and $KK(X; A, B)$ to obtain the following exact sequence:

$$\varinjlim \text{KK}(X; A_n, B) \to \text{KK}(X; A, B) \to \varinjlim \text{KK}(X; A_n, B).$$

By Lemma 6.6

$$\varinjlim \text{KK}(X; A_n, B) \cong \varinjlim \text{Hom}_{C(X, A)}(K(A_n), K(B)) 
\cong \text{Hom}_{C(X; A)}(\varinjlim K(A_n), K(B)) \cong \text{Hom}_{C(X; A)}(K(A), K(B)).$$
Using again Lemma 6.6 and Lemma 6.4, we get
\[
\lim_{\longleftarrow}^1 \text{KK}_1(X; A_n, B) \cong \lim_{\longleftarrow}^1 \text{Hom}_{C(X; \Lambda)}(\text{K}(A_n), \text{K}(SB)) \\
\cong \text{Ext}_{C(X; \Lambda)}(\text{K}(A), \text{K}(SB)).
\]

**Remark 6.12.** If \( A \) is a separable nuclear continuous \( C(X) \)-algebra with all the fibres in \( B \), then \( A \in B_k(X) \), and Theorem 6.11 shows that the exact sequence from Theorem 6.11 holds with \( \text{KK}(\Lambda) \) replacing \( E(X; A, B) \) replacing \( E(X; A, B) \).

For abelian groups \( G \) and \( H \), \( \text{PEExt}_Z(G, H) \) denotes the subgroup of \( \text{Ext}_Z(G, H) \) generated by pure extensions, that is, extensions \( H \to E \to G \) whose restrictions to all finitely generated subgroups of \( G \) split. Theorem 6.11 is a generalisation of the main result of [11], which corresponds to the case when \( X \) reduces to a point.

**Proposition 6.13.** Let \( A \) and \( B \) be separable \( C^* \)-algebras. If \( A \in B \), there is a natural isomorphism \( \text{Ext}_A(\text{K}(A), \text{K}(B)) \cong \text{PEExt}_Z(\text{K}(A), \text{K}(B)) \).

**Proof.** Consider the natural restriction map
\[
\eta: \text{Ext}_A(\text{K}(A), \text{K}(B)) \to \text{Ext}_Z(\text{K}(A), \text{K}(B)).
\]
Let \( \text{K}(B) 
\xrightarrow{\eta} 
\text{K}(A) 
\xrightarrow{\beta_n} 
\text{K}(A) 
\cong 
\text{K}(A),
\]
where \( \beta_n \in \Lambda \). Let \( \hat{y} \in M_n^{i+1} \) be a lifting of \( y \). Then the image \( \hat{x} := \beta_n(\hat{y}) \in M^0 \) of \( \hat{y} \) is a lifting of \( x \) of order \( n \). Thus the image of \( \eta \) is contained in \( \text{PEExt}_Z(\text{K}(A), \text{K}(B)) \).

Conversely, if \( \text{K}(B) \to G \to \text{K}(A) \) is a pure extension of \( Z/2 \)-graded abelian groups, then the UCT provides a separable \( C^* \)-algebra \( E \) and an extension of \( \text{C}^* \)-algebras \( B \otimes K \to E \to A \) such that \( \text{K}(B) \to \text{K}(E) \to \text{K}(A) \) is isomorphic to the given extension. We claim that \( \text{K}(B) \to \text{K}(E) \to \text{K}(A) \) is an extension of \( \Lambda \)-modules. Purity yields extensions
\[
\text{Tor}_q(\text{K}(A), Z/n) \to \text{Tor}_q(\text{K}(E), Z/n) \to \text{Tor}_q(\text{K}(A), Z/n)
\]
for any \( n \in \mathcal{P} \) and for \( q = 0, 1 \). Furthermore, there is a natural extension
\[
\text{Tor}_0(\text{K}(A), Z/n) \to \text{K}(A; Z/n) \to \text{Tor}_1(\text{K}(A), Z/n),
\]
and the same for \( E \) and \( B \). Now a diagram chase shows that \( \text{K}(B; Z/n) \to \text{K}(E; Z/n) \to \text{K}(A; Z/n) \) is an extension.

Having identified the image of \( \eta \) as \( \text{PEExt}_Z(\text{K}(A), \text{K}(B)) \), it remains to show that \( \eta \) is injective. We may assume that \( A \) is nuclear. Suppose that the extension \( \text{K}(B) \to \text{K}(E) \to \text{K}(A) \) splits. By the UCT, the class of the extension \( B \otimes K \to E \to A \) in \( \text{KK}_1(A, B) \) is zero. It follows that the extension \( B \otimes K \to E \to A \) is stably split, so that the extension \( \text{K}(B) \to \text{K}(E) \to \text{K}(A) \) is trivial. \( \square \)

The following example adapted from [10] shows that the map \( E(X; A, B) \to \text{Hom}_{C(X; \Lambda)}(\text{K}(A), \text{K}(B)) \) is not always surjective.

**Example 6.14.** Let \( X = \mathbb{N} \cup \{ \infty \} \) be the one-point compactification of \( \mathbb{N} \). We shall exhibit two separable continuous \( C(X) \)-algebras \( E_k \) and \( E_{k'} \) with all fibres isomorphic to Kirchberg algebras in the bootstrap category such that \( E_k \) and \( E_{k'} \)
have isomorphic filtrated K-theory but non-isomorphic filtrated K-theory with coefficients.

Let \( A \) be a Kirchberg algebra in the bootstrap category with \( K_0(A) = 0 \) and \( K_1(A) = \mathbb{Z}/n \) for \( n \geq 2 \). For \( k \in \mathbb{Z}/n \) let \( \varphi_k : A \to \mathcal{O}_\infty \) be a *-homomorphism such that \( [\varphi_k] = k \in KK(A, \mathcal{O}_\infty) \cong \mathbb{Z}/n \). Consider the \( C(\cdot) \)-algebra

\[
E_k = \{(f, a) \in C(X, \mathcal{O}_\infty) \oplus A \mid f(\infty) = \varphi_k(a)\}.
\]

We note that \( K_*(E_k) \cong K_*(E_{k'}) \) as \( C(X, \mathbb{Z}) \)-modules for any \( k, k' \), and we claim that if \( k\mathbb{Z}/n \neq k'\mathbb{Z}/n \), then \( K(E_k) \not\cong K(E_{k'}) \) as \( C(X, \Lambda) \)-modules. Indeed, \( K_0(E_k) = K_0(E_{k'}) = C_0(X, \mathbb{Z}) \) with \( C(X, \mathbb{Z}) \)-module structure \( fm = f(\infty)m \) for \( m \in \mathbb{Z}/n \). On the other hand,

\[
K_0(E_k; \mathbb{Z}/n) = \{(f, r) \in C(X, \mathbb{Z}/n) \oplus \mathbb{Z}/n \mid f(\infty) = kr\}.
\]

The coefficient map \( \rho : K_0(E_k) \to K_0(E_k; \mathbb{Z}/n) \) is \( g \mapsto (\hat{g}, 0) \). The Bockstein map \( \beta : K_0(E_k; \mathbb{Z}/n) \to K_1(E_k) \) is \( \beta(f, r) = r \).

Suppose that \( \alpha : K(E_k) \to K(E_{k'}) \) is an isomorphism of \( C(X, \Lambda) \)-modules. Then \( \alpha \) must act on \( K_0 \) by multiplication by a function \( u : X \to \{-1, 1\} \). Since \( \alpha \) is \( C(X, \mathbb{Z}) \)-linear and commutes with \( \rho \) and \( \beta \), there is a unit \( v \in \mathbb{Z}/n \) such that \( \alpha(k) = k\mathbb{Z}/n \to K_0(E_{k'}; \mathbb{Z}/n) \) is given by \( \alpha(f, r) = (uf, vr) \). Choose \( f \) such that \( (f, 1) \in K_0(E_k) \). It follows that for all sufficiently large \( i \) we have \( u(i)f(i) = k'v \) and hence \( \pm kr = k'v \). Thus \( k\mathbb{Z}/n = k'\mathbb{Z}/n \).

Next we generalise the previous example, constructing a suitable continuous \( C(\cdot) \)-algebra over any compact Hausdorff space \( X \).

**Example 6.15.** Let \( X \) be an infinite metrisable compact space. We shall exhibit two unital separable continuous \( C(\cdot) \)-algebras \( F \) and \( F' \) with all fibres isomorphic to Kirchberg algebras in the bootstrap category such that \( F \) and \( F' \) have isomorphic filtrated K-theory but non-isomorphic filtrated K-theory with coefficients.

Using the assumption on \( X \) we find a sequence \( (x_i)_{i=1}^\infty \) of distinct elements of \( X \) which converges to some \( x_\infty \in X \). Fix an embedding \( \mathcal{O}_\infty \subset \mathcal{O}_2 \). For each \( k \in \mathbb{Z}/n \) let \( A \) and \( \varphi_k : A \to \mathcal{O}_\infty \) be as in Example 6.14 Consider the \( C(\cdot) \)-algebra

\[
F_k := \{(f, a) \in C(X, \mathcal{O}_2) \oplus A \mid f(x_i) \in \mathcal{O}_\infty \forall i \in \mathbb{N}, f(x_\infty) = \varphi_k(a)\}.
\]

Choose \( k, k' \in \mathbb{Z}/n \) such that \( k\mathbb{Z}/n \neq k'\mathbb{Z}/n \) and set \( F = F_k \) and \( F' = F_{k'} \). Then \( F \) and \( F' \) have non-isomorphic filtrated K-theory with coefficients since their restrictions to the subspace \( Y := \{x_\infty\} \cup \{x_i \mid i \in \mathbb{N}\} \) are isomorphic to the \( C(Y) \)-algebras \( E_k \) and \( E_{k'} \) from Example 6.14 respectively. At the same time, we have an exact sequence of \( C(X) \)-algebras \( G \to F_k \to E_k \) with \( G = C_0(X \setminus Y, \mathcal{O}_2) \). Since \( K_*(\mathcal{O}_2) = 0 \), we see that \( K_*(G(T \setminus Y)) = 0 \) for all locally closed subsets \( T \) of \( X \). It follows that the filtrated K-theory of \( F \) is isomorphic to the filtrated K-theory of \( F' \) since we have seen that \( E_k \) and \( E_{k'} \) have this property.

**References**


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