The focus of the talk will be $C^*$-algs over a locally compact Hausdorff $X$. We recall the definition.

Def: (Kasparov) A $C^*$-alge A is a $C_0(X)$-algebra if there exists a homomorphism \( \Theta: C_0(X) \rightarrow L(M(A)) \) such that

\[ \Theta(C_0(X))A = A \]

The homomorphism $\Theta$ is called the structure homomorphism. We usually suppress $\Theta$ from the notation and write $fa$ for $\Theta(f)a = a\Theta(f)$ for $f \in C_0(X)$ and $a \in A$.

If $U$ is an open subset of $X$, we write $A(U) = C_0(U)A$ and if $F$ is closed in $X$, then $A(F) = A/A(U\setminus F)$.

When $F = \{x\}$, we write $A(x)$ for $A(F)$ and call this the fiber over $x$.

**Corollary.** There is an inclusion

\[ A \rightarrow \prod_{x \in X} A(x) \]

sending $a \in A$ to $(a(x))_{x \in X}$. It is a fact that for all $a \in A$, the map $x \mapsto \|a(x)\|$ is upper semicontinuous. (Notice that there is no map if we omit the norm since $a(x)$ lives in different $C^*$-algs for different $x \in X$.)
Defn: A is a continuous \((c^*_{(x)})\)-alg. (also called 
closed field \(c^*\text{-alg}\)) if \(\ x \rightarrow \|a(x)\| \) is continuous.

It is a fact that if \(A\) is separable and \(Prim(A)\) is 
Hausdorff, then \(A\) is a closed field over \(x = Prim(A)\).

The following is an example of a \((c^*_{(x)})\)-alg which 
is not a closed field.

Ex: Take \(A = \mathbb{C}\) as a \(c([0,1])\)-alg, with the 
action given by \(f \cdot a = f(x) \cdot a\).

The function \(x \rightarrow \|a(x)\|\) is hence not cts because 
it jumps at 1.

For now, we will restrict our attention to separable 
exact \(c^*\)-algebras which are continuous fields. (Exact 
means that \(A\) is a subalg of a nuclear \(c^*\text{-alg}\) by 
Kirchberg's Thm)

Theorem: (Blanchard - Kirchberg)

If \(A\) is a sep. exact continuous field over a compact 
space \(X\), then there exists an injective homomorphism 
equivariant \((c^*\text{-linear})\):

\[ A \rightarrow C(X, \mathbb{O}_2) \rightarrow C(X, L(H)) \]

So the algebras we are working with are generic \((c^*\text{-})\) 
invariant subalgebras of \(C(X, \mathbb{O}_2)\).
Example 1:

\[ A = \{ f \in C([0,1], M_3) : f(1/2) \in C^{*} \} \]

If \( A \) is as in the above theorem, there is an embedding

\[ A \hookrightarrow C(\mathbb{R}, L(M)) \]

and if \( U, F \subseteq A \) are open and closed respectively, then one can show that the above embedding respects restriction/inclusion in the sense that

\[ \begin{array}{c}
A(U) \hookrightarrow C_0(U, L(M)) \\
\downarrow \quad \downarrow \\
A \hookrightarrow C(\mathbb{R}, L(M)) \\
\downarrow \quad \downarrow \\
A(F) \hookrightarrow C(F, L(M))
\end{array} \]

is commutative.

Definition: We say that \( A \) is locally trivial if for each \( x \in X \) there exists \( U \subset X \) open with \( x \in U \) and s.t.

\[ A(U) \cong C_0(U, D) \quad \text{for some } C^*\text{-alg } D \]

Notice that the set field in Ex 1 is not locally trivial since \( f(1/2) \in C^* \) but \( A(x) \cong M_3 \) for \( x \neq 1/2 \). (Fibers should be locally isomorphic to \( A \) is loc. trivial).

Example 2: Let \( A_1 \) be the example above, and set

\[ A_2 = A_1 \otimes K. \]

Is \( A_2 \) locally trivial?
Example 3. \( A_3 = A_4 \otimes M_2 \). Again all the fibers are isomorphic to \( M_3 \). Is \( A_3 \) locally trivial?

Example 4. \( A_4 = A_1 \otimes \mathbb{C} \otimes \mathbb{O}_3 \), fibers \( \mathbb{C} \otimes \mathbb{O}_3 \).

Example 5. \( A_5 = A_1 \otimes \mathbb{C} \otimes \mathbb{O}_4 \), fibers \( \mathbb{C} \otimes \mathbb{O}_4 \).

It is not clear yet whether these are loc. trivial. We will use \( k \)-theory to answer these questions.

Let’s use \( k \)-thy to show that Ex 4 is not locally trivial. Consider the extension

\[
0 \to A \left( [0,1] \times \mathbb{Q}_2 \right) \to A \to A(\mathbb{Q}_2) \to 0
\]

which is just

\[
0 \to C_0 \left( [0,\infty) \right) \otimes M_3 \to A \to 0 \to 0
\]

\[
\oplus C_0 \left( \left( \frac{1}{2}, 1 \right) \right) \otimes M_3
\]

Since cones are contractible, they are trivial on \( k \)-thy and we get an isomorphism \( K_0(A) \cong K_0(C) \cong \mathbb{Z} \), with generator given by \( 1_A \). On the other hand, the map

\[
K_0(A) \to K_0(A(\mathbb{Q}_2))
\]

\[
\mathbb{Z}[\mathbb{Q}_2] \to K_0(M_3) \cong \mathbb{Z}
\]

is multiplication by 3, so in particular not surjective. Local triviality would imply that \( A \to A(\mathbb{Q}_2) \) induces an isomorphism on \( k \)-theory for all \( x \in \mathbb{Q}_2 \). Hence Example 4 is not loc. trivial.
Since tensoring with $V$ doesn't change the $K$-theory, it follows that Ex 2 is not loc. trivial either.

We will deal with Ex's 3-4-5 at the same time. Let $A \in \text{the cts field}$

\[ A = \left\{ f \in C([0,1], D) : f(1/2) \in \gamma(D) \right\} \]

where $D$ is a $C^*$-alg and $\gamma : D \to D$ is an injective $*$-homom. (Hence $\gamma(D)$ is isomorphic to $D$ but it is concretely a different $C^*$-alg). Consider the extension

\[ 0 \to CD \otimes CD \to A \to \gamma(D) \otimes D \to 0 \]

\[ f \mapsto f(1/2) \]

which induces an isomorphism $K_0(A) \xrightarrow{\cong} K_0(D)$. We wish to describe

\[ K_0(A) \xrightarrow{(\pi_2)_*} K_0(A \otimes K) \]

\[ \xrightarrow{\cong} K_0(D) \]

It follows that if $\gamma_*$ is not bijective, then $A$ is not locally trivial.

In Example 5, the $K$-thy of the fibers is $K_0(V_0)=\mathbb{Z}_3$ and the map $\gamma_*$ is multiplication by 3, which is the zero map. Hence it is not locally trivial.

In Examples 3 and 4, the maps are

\[ \mathbb{Z} \left[ \frac{1}{3} \right] \xrightarrow{\cdot 3} \mathbb{Z} \left[ \frac{1}{3} \right] \quad \mathbb{Z}_2 \xrightarrow{\cdot 3} \mathbb{Z}_2 \]
and both these maps are bijective, so there is no clear obstruction. It turns out that Exs. 3 and 4 are true but this requires a proof.

(Since $[0,1]$ is contractible, $A$ is loc. trivial iff $A$ is trivial.)

Observation: $A_2 \cong C([0,1], D)$ iff

$\exists$ a path $[0, 1/2] \to \text{End}(D)$, $t \mapsto \theta_t$, such that

$$
\begin{cases}
\theta_t \in \text{Aut}(D) & \text{if } 0 < t < 1/2 \\
\theta_{1/2} = \delta
\end{cases}
$$

(One can use the same kind path on the other half)

The isomorphism is then

$$
C([0,1/2]) \otimes D \to A([0,1/2])
$$

$f(t) \mapsto \theta_t(f(t))$

Remark: If $A_2$ is trivial, then $\delta$ must be a $KK$-equivalence since it is homotopic to an automorphism.

The converse is true if $D$ is a stable Kirchberg algebra. This is true by a result of Kirchberg and Phillips that states that a full endomorphism $D \to D$ is homotopic to an automorphism $KK$-trivial.

Let $A$ be a continuous field over $[0,1]$ and assume that $A([0,1]) \cong C([0,1]) \otimes D$ and $A(0) \cong D$. We would like to show that $A$ has the form $A_2$ for some $\delta$, except that now we can't expect $\delta$ to be an endomorphism, just an asymptotic morphism, for some unital $C^*$-algebra $D$. 
Consider the extension
\[ 0 \to C_0([0,1]) \otimes D \to A \to A(1) \to 0. \]
Its Bosby invariant is the map
\[ \sigma : D \to M(1)/I = C_0([0,1], D) / C_0([0,1], D) \]
Thus \( \sigma \) is an asymptotic homeomorphism \( \sigma = (\sigma_x)_{x \in [0,1]} \) from \( D \) to itself. Our aim shows that \( A \) is isomorphic to \( A_\sigma \), where
\[ A_\sigma = \{ (f, d) \in C_0([0,1], D) \otimes D : \lim_{x \to 1} \| f(x) - \sigma_x(d) \| = 0 \}. \]
In this way, asymptotic isomorphisms are in one-to-one correspondence with \( C([0,1]) \)-algebras with a singularity at \( x = 1 \).
The case \( \sigma_x = \text{constant equal to } \delta \) corresponds to \( A_\delta \). This is due to Connes-Higson.
One can easily produce finitely many singular pts, but how many can one produce? We will show an example of a $C([0,1])$-continuous field where all the fibers are isomorphic and every point in $[0,1]$ is singular.

Example: (Dadarlat-Elliott).

Let $D$ be a unital UCT Kirchberg algebra such that $K_0(D) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $[1_D] = (1,0)$. Choose a dense sequence $(x_n)_{n \in \mathbb{N}}$ in $[0,1]$. Let $\gamma$ be an endomorphism of $D$ such that

$$K_0(\gamma): \mathbb{Z}^2 \to \mathbb{Z}^2 \text{ is given by } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Let $A_n = \{ f \in C([0,1],D) : f(x_n) \in \gamma(D) \}$, and set

$$B = \bigotimes_{n=1}^{\infty} A_n$$

Balanced tensor product of $C(\mathbb{R})$-algebras.

It gives a $C(\mathbb{R})$-alg or opposed to a $C(\mathbb{R} \times \mathbb{R})$-alg.

It is easy to see that $B(\mathbb{R}) \cong \bigotimes D$, but every point in $[0,1]$ is singular. Indeed, for every $(a,b) \in [0,1]$ there exists $x \in (a,b)$ s.t.

$$(\pi_x)_*: K_0(b) \to K_0(b(x)) \text{ is not injective.}$$

Hence $B$ is nowhere trivial, and all points are singular.

Remark: anycts field with isomorphic fibers that are finitely dimensional is necessarily $0$-trivial.

Semiprojective $C^*$-algs are crucial in describing the structure of its fields.
1. Pullbacks

Any continuous field with stable Kirchberg algebras as fibers, s.t. moreover $K_i$ of the fibers is torsion free, is an inductive limit of continuous fields with finitely many singular points. One then looks at the asymptotic gluing morphisms to treat these C* fields.

Application: Any C* field over $\mathbb{T}^n$, with fiber $G\times K$ is trivial. Prove: Approximate it by C* fields with finitely many singularities. These must be trivial because any endomorphism of $G\times K$ is KK-equivalent to an automorphism (no K-theory), and thus the original C* field being a limit of trivial fields, is trivial itself.

$n$-Pullbacks: Let $X$ be $\mathbb{T}^n$, $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ closed subsets. For $j = 0, \ldots, n$, let $E_j$ be a trivial field over $Y_j$. Suppose we are given $n$ monomorphisms $\delta_{ij}: E_i |_{Y_i \cap Y_j} \to E_j |_{Y_i \cap Y_j}$, $(Y_i \cap Y_j)$-linear, such that $(\delta_{jk})_{x_0} \circ (\delta_{ij})_{x_0} = (\delta_{ik})_{x_0}$ for $x \in Y_i \cap Y_j \cap Y_k$. The pullback is then

\[ A = \{ \left( e_{i_1}, \ldots, e_{i_n} \right) \in E_0 \otimes \cdots \otimes E_n : \quad \delta_{ij}(e_i) = (Y_j)_x(e_{i_2}) \} \]

lego fields.
Theorem: Let $A$ be a separable nuclear C*-field over a space $X$ of dimension $n < \infty$. Suppose each $A_{i} = K(A_{i}(x))$ is torsion free. Then $\exists A, A_{1}, A_{2}, \ldots A_{n}$ such that $A_{i} = A_{i-1} \oplus A_{i+1}$, with $c(x)$-linear inclusions and such that each $A_{i}$ is an $n$-pull-back of tor. trivial fields.

Corollary: If $A$ is a separable nuclear C*-field such that each fiber satisfies the UCT, then $A$ itself satisfies the UCT.

Proof: Show that every such field is $KK(x)$-equivalent to a Kirchberg field and approximate this $A$ using the above.

Theorem: Let $X$ be a compact Hausdorff space of finite dimension. Suppose that $A$ is a cts field over $X$ such that $A(x) \leq 10_{x} \otimes K$ for all $x \in X$. Then

$$A \simeq c(x) \otimes K \otimes K.$$ 

The point is that the gluing maps $\Omega_{x} : Y_{x} \otimes \text{End}(10_{x})$ are $KK$-equivalent to automorphisms, and thus each of the pull-backs is trivial.

Nowadays there are proofs of this result that use the classification developed by Kirchberg. See some work by Higson-Rørdam-Winter where they show that if the same assumption for the fibers of the fibers absorb $10_{x}$, then the field itself absorbs $0_{2}$. 