Plan of this talk

Examples of abelian categories
Equivalence of categories
Exact sequences and exact functors
Adjunctions
Projective generators
Morita equivalence
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Examples of Abelian Categories

- $\mathbf{R}$ - a unital ring.
  - $\mathbf{Mod}(\mathbf{R})$ - the category of left $\mathbf{R}$-modules.
- $\mathbf{G}$ - an $\ell$-group.
  - $\mathbf{Rep}(\mathbf{G})$ - the category of smooth complex representations of $\mathbf{G}$.
- $\mathbf{A}$ - an idempotented algebra.
  - $\mathbf{Mod}(\mathbf{A})$ - the category of non-degenerate left $\mathbf{A}$-modules.
- $\mathbf{X}$ - a topological space.
  - $\mathbf{Sh}(\mathbf{X})$ - the category of sheaves of complex vector spaces over $\mathbf{X}$. 
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Examples of idempotented algebras

- \( \mathcal{L} \)-space.
- \( \mathcal{S}(\mathcal{L}) \) - the algebra of smooth, compactly supported complex functions on \( \mathcal{L} \) with respect to pointwise multiplication.
- \( \mathcal{G} \)- an \( \mathcal{L} \)-group.
- \( \mathcal{H}(\mathcal{G}) \) - the algebra of smooth, compactly supported measures on \( \mathcal{G} \) with respect to convolution.
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Equivalence of categories

Definition
An equivalence between $\mathcal{A}$ and $\mathcal{B}$ is the data of two functors, $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$, and isomorphisms of functors $\eta: \text{Id}_\mathcal{B} \cong F \circ G$ and $\nu: \text{Id}_\mathcal{A} \cong G \circ F$.

Let $F: \mathcal{A} \to \mathcal{B}$ be a functor. When is it an equivalence of categories (i.e. when can we find $G, \eta, \nu$ as above)?

Proposition $F: \mathcal{A} \to \mathcal{B}$ is an equivalence i.f.f. it is fully faithful and essentially surjective.

fully faithful: $\text{Hom}(X, Y) \to \text{Hom}(FX, FY)$ is an isomorphism, for all $X, Y \in \mathcal{A}$.

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For abelian categories - functors should be additive.
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For abelian categories - functors should be additive.
Equivalences of categories: Galois and Poincaré

Galois

$K \subset K$ - a field (say of char. 0), equipped with an embedding into an algebraic closure.

$\mathbb{G} = \text{Gal}(K/K)$ - the profinite Galois group.

$\text{Fin}_K$ - the category of finite extension fields of $K$.

$\text{Sets}_G$ - the category of finite transitive continuous $G$-sets.

Theorem \[ E \to \text{Hom}_K(E, K) \] provides an equivalence of categories $\text{Fin}_K \approx \text{Sets}_G$.

Poincaré

$(X, x_0)$ - a pointed connected (nice) topological space.

$\mathbb{G} = \pi_1(X, x_0)$.

$\text{Cov}_X$ - the category of covering spaces of $X$.

$\text{Sets}_G$ - the category of $G$-sets.

Theorem Sending a covering space to its fiber over $x_0$ provides an equivalence of categories $\text{Cov}_X \approx \text{Sets}_G$. 

Background From Category Theory
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\( K \subset \overline{K} \) - a field (say of char. 0), equipped with an embedding into an algebraic closure. \( G = Gal(\overline{K}/K) \) - the profinite Galois group. \( Fin_K \) - the category of finite extension fields of \( K \).
Equivalences of categories: Galois and Poincare

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$K \subset \overline{K}$ - a field (say of char. 0), equipped with an embedding into an algebraic closure. $G = Gal(\overline{K}/K)$ - the profinite Galois group. $\mathbf{Fin}_K$ - the category of finite extension fields of $K$. $\mathbf{Sets}_G$ - the category of finite transitive continuous $G$-sets.
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Equivalences of categories: Gelfand and BZ

- **Gelfand**
  - Comm - the category of commutative $C^*$-algebras.
  - Top - the category of compact topological spaces.

  **Theorem** $\text{Comm} \approx \text{Top}^{\text{op}}$ (l.t.r. - forming spectrum, r.t.l. - forming algebra of continuous functions w. sup. norm).

- **Bernstein-Zelevinsky**
  - Proposition
    - For an $\ell$-space $X$ we have $\text{Sh}(X) \approx \text{Mod}(S(X))$.
    - For an $\ell$-group $G$ we have $\text{Rep}(G) \approx \text{Mod}(H(G))$.

Background From Category Theory
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Exact functors

A \rightarrow B \rightarrow C is exact (in B) if \text{Im}(f) = \text{Ker}(g).

0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 is exact if it is exact at all places. This is called a short exact sequence.

Interpretation: A is a subobject of B and C is a quotient of B, and B is “glued” from A and C.

Exact functor: maps s.e.s. to s.e.s.

Left exact functor: only resulting 0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) stays exact.

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Example

Let $M \in A$. Then the functor $H_M : A \to A$ given by $H_M(N) = \text{Hom}(M, N)$ is left exact.

Definition

$M$ is called projective if $H_M$ is exact.

In $\text{Mod}(R)$ ($R$ unital) an object is projective i.f.f. it is a direct summand of a free module.
Example

Let $M \in \mathcal{A}$. Then the functor $H_M : \mathcal{A} \to \mathcal{Ab}$ given by

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Exact functors: examples

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Adjunction - definition

**Definition**

\[ \text{A} \to \text{B} \text{ is left adjoint to } \text{G} : \text{B} \to \text{A} \]\n
if for any \( \text{X} \in \text{A} \), \( \text{Y} \in \text{B} \) we are given an isomorphism \( \alpha_{\text{X,Y}} : \text{Hom}_B(\text{FX}, \text{Y}) \to \text{Hom}_A(\text{X}, \text{GY}) \) which is functorial in \( \text{X} \) and \( \text{Y} \).

**Definition**

\[ \text{A} \to \text{B} \text{ is left adjoint to } \text{G} : \text{B} \to \text{A} \] if we are given \( \alpha : \text{Id}_A \to \text{G} \circ \text{F} \) and \( \beta : \text{F} \circ \text{G} \to \text{Id}_B \) such that \( \text{F} \to \text{F} \circ \text{G} \circ \text{F} \to \text{F} \) and \( \text{G} \to \text{G} \circ \text{F} \circ \text{G} \to \text{G} \) are the identity maps.

**Theorem**

Between any two left adjoints of a functor \( \text{G} \) there is a unique isomorphism which respects the adjunction data.

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Background From Category Theory
Definition

$F : \mathcal{A} \to \mathcal{B}$ is left adjoint to $G : \mathcal{B} \to \mathcal{A}$ if for any $X \in \mathcal{A}$, $Y \in \mathcal{B}$ we are given an isomorphism

$\alpha_{X,Y} : \text{Hom}_\mathcal{B}(FX, Y) \to \text{Hom}_\mathcal{A}(X, GY)$ which is functorial in $X$ and $Y$. 

Theorem

Between any two left adjoints of a functor $G$ there is a unique isomorphism which respects the adjunction data.
Adjunction - definition

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Between any two left adjoints of a functor $G$ there is a unique isomorphism which respects the adjunction data.
Adjunction - examples

**Principle:** Category theory gives a new, unifying, construction tool - the adjoint of a functor.

**Example:**
- **A** - metric spaces, **B** - complete metric spaces.
  - **G**: $B \to A$, the embedding functor.
  - It admits a left adjoint. It is called the completion of a metric space.

**Example:**
- **A** - sets, **B** - $\mathbb{R}$-modules.
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Adjunction - further examples

Example

A - abelian groups, $B - \mathbb{R}$-modules. Fix $M \in B$. $G : B \to A$ sending $N$ to $\text{Hom}(M, N)$. It admits a left adjoint. It is called tensor product construction: $V \mapsto M \otimes \mathbb{Z}V$.

Example

$H \subset G$ - finite groups. $A = \text{Rep}(H)$, $B = \text{Rep}(G)$. $G : B \to A$ the forgetful functor. It admits a left adjoint. It is called induction (this is called Frobenius reciprocity).

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Properties of adjoint functors

Theorem
Let $F : A \to B$ be a left adjoint. Then $F$ is right exact and commutes with direct sums.

A kind of converse is:
Theorem
Let $R$ be unital and $B$ an abelian category. Suppose that $F : \text{Mod}(R) \to B$ is right exact and commutes with direct sums. Then $F$ admits a right adjoint.

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A third definition of equivalence of categories is: an adjunction, such that the unit and counit maps are isomorphisms.
A projective generator is an abelian category with direct sums. Let $P \in A$. Note that $\text{End}(P)$ is a unital ring. Set $R := \text{End}(P)^{\text{op}}$. For every $M \in A$, $\text{Hom}(P, M)$ is naturally an $R$-module. We get a functor $\Phi : A \to \text{Mod}(R)$ in this way (it “lifts” the functor $H_P$ from before).

When is this functor an equivalence? This question will teach us how to identify abstract abelian categories as module categories.

**Theorem** $\Phi$ is an equivalence i.f.f. $P$ is a compact projective generator.

$P$ is projective means that $H_P$ is exact. $P$ is generator means that whenever $M \neq 0$ we have $H_P(M) \neq 0$. $P$ is compact means that $H_P$ commutes with direct sums.
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Examples

Consider $\text{Mod}(R)$ ($R$ unital).

Let $P = R^n$. Then $P$ is a compact projective generator and $\text{End}(P) = M_{n}(R)$.

We get an equivalence $\text{Mod}(R) \approx \text{Mod}(M_{n}(R))$.

Example

Consider $\text{Mod}(A)$ ($A$ idempotented).

Let $e \in A$ be an idempotent, and set $P := Ae$.

Then $P$ is compact projective and $\text{End}(P) = eAe$.

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Consider $\text{Mod}(R)$ ($R$ unital).

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Examples - continuation

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- unital. Let $A$ be the ring of all $\mathbb{N} \times \mathbb{N}$ matrices, whose all entries, except finitely many, are 0. Let $e \in A$ be the element whose $(1, 1)$-entry is 1, and all else are 0. Then $A = AeA$, and $\text{End}(Ae)^{\text{op}} \cong R$. Hence, we get $\text{Mod}(A) \cong \text{Mod}(R)$.
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Morita equivalence

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There are, for a unital ring, some properties that are stable under Morita equivalence.

Example

The center of \( R \) is definable as an object attached to the abelian category \( \text{Mod}(R) \), and thus is a construction which is stable under Morita equivalence. Indeed, given an abelian category, define its center to be the endomorphism ring of the identity functor.
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