Local Langlands correspondence and examples of ABPS conjecture

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23/08/2013
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Notation

- $F$ non-archimedean local field: finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$
- $O_F = \{x \in F, \nu(x) \geq 0\}$, ring of integers of $F$
- $\mathfrak{p}_F = \{x \in F, \nu(x) > 0\}$, unique maximal ideal of $O_F$
- $k_F = O_F/\mathfrak{p}_F \simeq \mathbb{F}_q$, residual field of $F$ (finite field)
- $\varpi_F \in O_F$ uniformizer of $F$
- $\overline{F}$ separable algebraic closure of $F$
• $K/F$ finite extension
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• $\wp_F O_K = \wp_K^e O_K$,
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$K/F$ finite Galois extension, $k_K/k_F$ is also Galois extension

$\forall \sigma \in Gal(K/F), \sigma(\mathcal{O}_K) = \mathcal{O}_K, \sigma(p_K) = p_K$

$$Gal(K/F) \longrightarrow Gal(k_K/k_F)$$
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$$
1 \longrightarrow I_{K/F} \longrightarrow Gal(K/F) \longrightarrow Gal(k_K/k_F) \longrightarrow 1
$$
Definition
An extension $K/F$ is unramified if the ramification index $e(K/F) = 1$ and $k_K/k_F$ is separable.
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Theorem
$\forall n \geq 1, \exists! F \subset F_n \subset \overline{F}$ unramified extension of degree $n$ $F_n/F$ is Galois, $k_{F_n} \simeq \mathbb{F}_{q^n}$ and $\text{Gal}(F_n/F) \xrightarrow{\sim} \text{Gal}(k_{F_n}/k_F)$
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$Gal(F_{ur}/F) \simeq \lim_{\leftarrow n \geq 1} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

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The absolute Galois group of $F$ is $\Gamma_F = Gal(\overline{F}/F)$
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where $E$ ranges over finite Galois extension such that $F \subset E \subset \overline{F}$
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The absolute Galois group of $F$ is $\Gamma_F = \text{Gal}(\overline{F}/F)$

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where $E$ ranges over finite Galois extension such that $F \subset E \subset \overline{F}$

Topology on $\Gamma_F$ : Open neighborhood basis at the identity are $\text{Gal}(\overline{F}/K)$ with $K/F$ finite extension

$\Gamma_F$ is a compact Haussdorff profinite group
Local Langlands correspondence and examples of ABPS conjecture

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\[ \text{Gal}(\overline{F}/F) \longrightarrow \text{Gal}(F_{ur}/F) \longrightarrow 1 \]
\[ 1 \rightarrow I_F \rightarrow Gal(\bar{F}/F) \rightarrow Gal(F_{ur}/F) \rightarrow 1 \]
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\[ 1 \rightarrow I_F \rightarrow \text{Gal}(\overline{F}/F) \rightarrow \text{Gal}(F_{ur}/F) \rightarrow 1 \]

\[ \langle \Phi \rangle \]
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\[
1 \longrightarrow I_F \longrightarrow \text{Gal}(\overline{F}/F) \longrightarrow \text{Gal}(F_{ur}/F) \longrightarrow 1
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\[
\mathbb{Z}
\]
1 \rightarrow I_F \rightarrow \text{Gal}(\overline{F}/F) \rightarrow \hat{\mathbb{Z}} \rightarrow 0

Definition
The Weil group of $F$ is the topological group, with underlying abstract group the inverse image in $\Gamma_F$ of $\langle \Phi \rangle$, such that:

- $I_F$ is an open subgroup of $W_F$
- the topology on $I_F$, as subspace of $W_F$, coincides with its natural topology as subspace of $\Gamma_F$
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W_F \rightarrow \mathbb{Z} \rightarrow 0
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- the topology on $I_F$, as subspace of $W_F$, coincides with its natural topology as subspace of $\Gamma_F$
Let $W_F^{der} = [W_F, W_F]$ and $W_F^{ab} = W_F / W_F^{der}$.

**Artin Reciprocity Map**

There is a canonical continuous group morphism

$$a_F : W_F \longrightarrow F^\times$$

with the following properties.
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- $x \in W_F$ is a geometric Frobenius, if and only if $a_F(x)$ is an uniformizer of $F$
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GL_1(F) & \longrightarrow GL_1(\mathbb{C})
\end{align*}
\]

If \( T \) is a split torus over \( F \), then

\[
\begin{align*}
\left\{ \text{irreducible representations of } T \right\} & \leftrightarrow \left\{ \text{admissible morphisms } W_F \longrightarrow \hat{T} \right\} \\
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(+ equality of \( L \)-functions)
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(+ equality of \( L \)-functions)
This is the local Langlands correspondence for \( GL(1) \) and for split tori.
For $\psi : W_F \to GL_n(\mathbb{C})$ we can define a $L$-function by

$$L(s, \psi) = det(1 - \psi(\Phi)|_{V^I_F}q^{-s})^{-1}$$
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$$L(s, \psi) = \text{det}(1 - \psi(\Phi)|_{V^l_F q^{-s}})^{-1}$$

Godement and Jacquet defined $L$-functions for the representations of $GL_n(F)$. For $n = 2$,

- for supercuspidal representations are 1
- for Steinberg representation (and their twist by character) involve one factor with $q^{-s}$
- for other involve two factors with $q^{-s}$
Definition
The Weil-Deligne group is $WD_F = \mathbb{C} \rtimes W_F$ where $W_F$ acts on $\mathbb{C}$ by $wxw^{-1} = |w|x$ for $w \in W_F, x \in \mathbb{C}$. 

A Langlands parameter for $GL(n)$ is a continuous morphism $\psi: WD_F \rightarrow GL_n(\mathbb{C})$ such that the restriction to $\mathbb{C}$ is algebraic, the image of $\mathbb{C}$ consists of unipotent elements and the image of $W_F$ consists of semisimple elements.

We can identify one Langlands parameter $\psi$ to a pair $(\rho, N)$ where $\rho: W_F \rightarrow GL_n(\mathbb{C})$ is a continuous morphism such that his image consists of semisimple elements and $N \in \text{gl}_n(\mathbb{C})$ is such that $\rho(w)N\rho(w)^{-1} = |w|N$ for all $w \in W_F$. 

$\psi((z, w)) = \exp(zN)\rho(w)$, $(z, w) \in WD_F$.
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$$\psi(z, w) = \exp(zN)\rho(w), \ (z, w) \in WD_F$$
For $w \in W_F$, let $h_w = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \in SL_2(\mathbb{C})$. 
For $w \in W_F$, let $h_w = \left( |w|^{1/2}, |w|^{-1/2} \right) \in SL_2(\mathbb{C})$.

There is an isomorphism between $\mathbb{C}$ and the subgroup $G_a$ of $SL_2(\mathbb{C})$ of the matrices $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{C}$.
For \( w \in W_F \), let \( h_w = \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix} \in SL_2(\mathbb{C}) \).

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The conjugation by \( h_w \) in \( G_a \) induce the same action as \( W_F \) on \( \mathbb{C} \), and extend to \( SL_2(\mathbb{C}) \).
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The conjugation by $h_w$ in $G_a$ induce the same action as $W_F$ on $\mathbb{C}$, and extend to $SL_2(\mathbb{C})$. We can consider the semidirect product $SL_2(\mathbb{C}) \rtimes W_F$ with the product

$$(x_1, w_1)(x_2, w_2) = (x_1h_{w_1}x_2h_{w_1}^{-1}, w_1w_2)$$
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This group is isomorphic to $SL_2(\mathbb{C}) \rtimes W_F$ via the map $(x, w) \mapsto (x h_w, w)$. 
We denote $W'_F = W_F \times SL_2(\mathbb{C})$. 
We denote $W'_F = W_F \times SL_2(\mathbb{C})$. If a Langlands parameter is identified to $(\rho, N)$, then it correspond to a morphism $\psi : W'_F \longrightarrow GL_n(\mathbb{C})$ with $\psi \left( 1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) = \exp(N)$ and $\psi(w, h_w) = \rho(w)$. 
Local Langlands correspondence (conjecture)

Let $G$ a split connected reductive group over $F$ and $\widehat{G}$ is Langlands dual. Let $\Pi(G)$ the set of isomorphism classes of irreducible representations of $G$ and $\Phi(G)$ the set of equivalence classes of admissible morphisms for $G$. There is finite-to-one map $\text{rec} : \Pi(G) \rightarrow \Phi(G)$ with these properties
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- $\pi \in \Pi(G)$ is essentially square-integrable, if and only if, the image of $W_F$ by $\phi_{\pi} : W_F \longrightarrow \widehat{G}$ is not contained in any proper Levi subgroup of $\widehat{G}$. 

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- \( \pi \in \Pi(G) \) is essentially square-integrable, if and only if, the image of \( W_F' \) by \( \phi_\pi : W_F' \rightarrow \hat{G} \) is not contained in any proper Levi subgroup of \( \hat{G} \).

- \( \pi \in \Pi(G) \) is tempered, if and only if, the image of \( W_F \) by \( \phi_\pi : W_F' \rightarrow \hat{G} \) is bounded
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- $\pi \in \Pi(G)$, $\chi$ a character of $F^\times$, then $rec(\pi \otimes \chi) = rec(\pi)\chi$. 

Equality of $L$-functions and $\varepsilon$-factors.
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- $\pi \in \Pi(G)$ is essentially square-integrable, if and only if, the image of $W'_F$ by $\phi_\pi : W'_F \rightarrow \hat{G}$ is not contained in any proper Levi subgroup of $\hat{G}$.
- $\pi \in \Pi(G)$ is tempered, if and only if, the image of $W_F$ by $\phi_\pi : W'_F \rightarrow \hat{G}$ is bounded
- $\pi \in \Pi(G)$, $\chi$ a character of $F^\times$, then $\text{rec}(\pi \otimes \chi) = \text{rec}(\pi)\chi$.
- Equality of $L$-functions and $\varepsilon$-factors.
Let $J = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & & \end{pmatrix}$ and $J' = \begin{pmatrix} 1 & & \\ & -1 & 1 \\ -1 & & \end{pmatrix}$.

For $M \in GL_4(F)$, $^tM = J^tMJ$.

Let $G = Sp_4(F) = \{ g \in SL_4(F) | ^t g J' g = J' \}$.
Let \( J = \begin{pmatrix} \cdots & 1 \\ 1 & \cdots \end{pmatrix} \) and \( J' = \begin{pmatrix} & & & 1 \\ & & -1 & 1 \\ -1 & & & \end{pmatrix} \).

For \( M \in GL_4(F) \), \( {}^\tau M = J^t MJ \).

Let
\[
G = Sp_4(F) = \{ g \in SL_4(F) \mid {}^t g J' g = J' \}
\]
and \( B = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & \\ * & * & * & \end{pmatrix} \) minimal parabolic subgroup of \( Sp_4(F) \).
Let $J = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $J' = \begin{pmatrix} & & & 1 \\ & & -1 & \\ -1 & & & \\ & & & -1 \end{pmatrix}$.

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Let $\quad G = Sp_4(F) = \{g \in SL_4(F)|^t gJ'g = J'\}$

and $B = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$ minimal parabolic subgroup of $Sp_4(F)$.

There are three parabolic subgroups of $Sp_4(F): B, P, Q$ with respective Lévi $T, M, L$. 
Local Langlands correspondence and examples of ABPS conjecture

Ahmed Moussaoui

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- $L = \left\{ \begin{pmatrix} t \\ g \\ t^{-1} \end{pmatrix}, \ t \in F^\times, \ g \in SL_2(F) \right\}, \ L \simeq GL_1(F) \times SL_2(F)$
The Weyl group of $G$ is $W = N_G(T)/T \cong S_2 \rtimes (\mathbb{Z}/2\mathbb{Z})^2$, generated by

$$a = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$
Action of $W$ on $T$ and on their irreducible representations

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<tr>
<th></th>
<th>$w \cdot (t_1, t_2)$</th>
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We consider two inertial classes $s = [T, \chi \boxtimes 1]$ and $t = [T, \chi \boxtimes \zeta]$. 
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The isotropy subgroup of $s$ is the subgroup of elements $w$ in $W$ such that there exists an unramified character $\phi \in \psi(T)$,

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For $s$ and $t$ we have $W_s = W_t = \{1, b\}$ and $T_s = T_t = \psi(T) \cong (\mathbb{C}^\times)^2$. 
The extended quotient is

\[ T_s/W_s = T_s/W_s \sqcup T_s^b/Z_{W_s}(b) \]
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\[ T_s \!//\! W_s = T_s / W_s \sqcup T_s^b / Z_{W_s}(b) \]

Let \((t_1, t_2) \in T_s\). We have

\[ b \cdot (t_1, t_2) = (t_1, t_2) \iff (t_1, t_2^{-1}) = (t_1, t_2) \iff t_2 = \pm 1. \]
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Remark: It is the same extended quotient for \(s\) and \(t\).
Theorem (Sally, Tadic)

Let $\psi_1, \psi_2$ two characters of $F^\times$. Then $\psi_1 \times \psi_2 \times 1$ is reducible, if and only if

- $\psi_1^{\pm 1} = \nu^{\pm 1} \psi_2$
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$\psi \times \zeta \rtimes 1 = \psi \rtimes T_1^\zeta + \psi \rtimes T_2^\zeta$, $T_i^\zeta$ tempered representation of $SL_2(F)$ ($\zeta$ character of $F^\times$ of order two).
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$\psi \times \nu \times 1$ has two subquotients: $\psi \times 1_{SL_2}$ and $\psi \times St_{SL_2}$
\[ s = [T, \chi \boxtimes 1] \]

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Remark: \((e, (z_1, z_2))\) and \((b, (z, 1))\) are in the same \(L\)-packet.
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\[ \mathfrak{s} = [T, \chi \boxtimes 1] \]

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\[ T_s // W_s = (T_s // W_s)_{u_0} \sqcup (T_s // W_s)_{u_\varepsilon} \]
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$(T_s / / W_s)_{u_e} = T_s^{b+} / W_s$,
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\[ T_s / W_s = (T_s / W_s)_{u_0} \sqcup (T_s / W_s)_{u_e} \]

\[(T_s / W_s)_{u_0} = T_s / W_s \sqcup T_s^{b-} / W_s, \text{ cocharacter } h_{u_0}(\tau) = (1, 1) \]
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Remark : \((e, (z, -1))\) and \((b, (z, -1))\) are in the same \(L\)-packet
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Local Langlands correspondence and examples of ABPS conjecture

Ahmed Moussaoui

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The Langlands dual group of $Sp_4(F)$ is $SO_5(\mathbb{C})$, with $SO_5(\mathbb{C}) = \{ g \in SL_5(\mathbb{C}), ^t g J g = J \}$ and the maximal torus

$$\tilde{T} = \left\{ \begin{pmatrix} z_1 & z_2 \\ z_2^{-1} & z_1^{-1} \end{pmatrix}, z_i \in \mathbb{C}^\times \right\} \simeq (\mathbb{C}^\times)^2$$
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The Lie algebra of $SO_5(\mathbb{C})$ is

$$\mathfrak{so}_5(\mathbb{C}) = \left\{ \begin{pmatrix} y_1 & x_{12} & x_{13} & x_{14} & 0 \\ x_{21} & y_2 & x_{23} & 0 & -x_{14} \\ x_{31} & x_{32} & 0 & -x_{23} & -x_{13} \\ x_{41} & 0 & -x_{32} & -y_2 & -x_{12} \\ 0 & -x_{41} & -x_{31} & -x_{21} & -y_1 \end{pmatrix}, x_{ij}, y_i \in \mathbb{C} \right\}$$
The Langlands parameters for \( \chi_1 \boxtimes \chi_2 \) is
\[
\rho(\chi_1, \chi_2) : \ W_F \rightarrow \widehat{T}
\]
\[
w \mapsto \begin{pmatrix}
\chi_1(w) & \chi_2(w) & 1 \\
\chi_2(w)^{-1} & \chi_1(w)^{-1}
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The Langlands parameters for the irreducible subquotients of $\psi_1 \chi \times \psi_2 \times 1$ are $(\rho(\psi_1 \chi, \psi_2), 0)$, $(\rho(\psi_1 \chi, \nu), 0)$ and $(\rho(\psi_1 \chi, \nu), N)$ with

$$N = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$
The Langlands parameters are also the morphisms from $W'_F$ to $SO_5(\mathbb{C})$. For $(\rho_\psi, \nu), N)$ the corresponding morphism is

$$\phi : W'_F \rightarrow \widehat{T}$$

$$(w, s) \mapsto \begin{pmatrix}
(\psi_1\chi)(w) \\
S_3(s) \\
(\psi_1\chi)(w)^{-1}
\end{pmatrix}$$

where $S_3$ is the irreducible representation of dimension 3 of $SL_2(\mathbb{C})$. 
The Langlands parameters are also the morphisms from $W'_F$ to $SO_5(\mathbb{C})$. For $(\rho(\psi_1\chi, \nu), N)$ the corresponding morphism is

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where $S_3$ is the irreducible representation of dimension 3 of $SL_2(\mathbb{C})$.

We have $\phi\left(1, \begin{pmatrix} \tau^{-1} \\ \tau \end{pmatrix}\right) = \\
\begin{pmatrix}
1 \\
\tau^{-2} \\
1 \\
\end{pmatrix}$
For the irreducible subquotient of $\psi_1 \chi \times \psi_2 \zeta \rtimes 1$ there are $(\rho(\psi_1 \chi, \psi_2 \zeta), 0)$. 
For the irreducible subquotient of $\psi_1 \chi \times \psi_2 \zeta \trianglerighteq 1$ there are $(\rho(\psi_1 \chi, \psi_2 \zeta), 0)$. We find in this way the cocharacters and the decomposition.
$s = [T, \chi \boxtimes 1]$

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\[
\mathcal{S} = [T, \chi \boxtimes 1]
\]

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\[ t = [T, \chi \boxtimes \zeta] \]

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Thank you for your attention.
Anne-Marie Aubert, Paul Baum, and Roger Plymen. Geometric structure in the representation theory of reductive $p$-adic groups II.


A. W. Knapp. Introduction to the Langlands program.