Structure of the Iwahori-Hecke Algebra

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$F$ - Non-archimedean local field.
$\mathcal{O}$ - ring of integers, $\mathfrak{p} = (\pi)$ - maximal ideal.
$k := \mathcal{O}/\mathfrak{p}$, $|k| = q$. 

$G$ - Split, connected, reductive group defined over $F$.
$A$ - split maximal torus $A$.
$B = AN$ - Borel subgroup containing $A$.

We assume that $G$, $A$, $N$ etc. are defined over $\mathcal{O}$. 
Notation

\( F \) - Non-archimedean local field.
\( \mathcal{O} \) - ring of integers, \( \mathfrak{p} = (\pi) \) - maximal ideal.
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\( G \) - Split, connected, reductive group defined over \( F \).
\( A \) - split maximal torus \( A \).
\( B = AN \) - Borel subgroup containing \( A \).
We assume that \( G, A, N \) etc. are defined over \( \mathcal{O} \).
$K := G(\mathcal{O})$
$W := N_{G(F)}(A(F))/A(F)$ - the finite Weyl group.
\[ K := G(0) \]
\[ \mathcal{W} := N_{G(F)}(A(F))/A(F) - \text{the finite Weyl group.} \]

We will sometimes write \( G, A \) for \( G(F), A(F) \) (respectively) for the sake of brevity.
The Iwahori subgroup and the Iwahori Hecke Algebra

Definition

The Iwahori subgroup $I$ of $K$ is the inverse image of $B(k)$ in $G(O)$ under the map $G(O) \to G(k)$. 

\[
H_I := \mathbb{C}[c(I\setminus G(F)/I)].
\]
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Definition

Define the Iwahori-Hecke algebra,

$$\mathcal{H}_I := C_c(I \setminus G(F)/I).$$
Finite dimensional Hecke algebra

\[ \mathcal{H}_f := C(B(k) \setminus G(k)/B(k)). \]

By Bruhat decomposition,

\[ \{ T_w := 1_{B(k)wB(k)} | (w \in W) \} \]

forms a basis of \( \mathcal{H}_f \).
Finite dimensional Hecke algebra

\[ \mathcal{H}_f := \mathbb{C}(B(k) \backslash G(k)/B(k)). \]

By Bruhat decomposition,

\[ \{ T_w := 1_{B(k)wB(k)} | (w \in W) \} \]

forms a basis of \( \mathcal{H}_f \).

**Proposition**

For \( f \in \mathbb{C}(B(k) \backslash G(k)/B(k)) \), we can define \( \tilde{f} \in \mathbb{C}_c(I \backslash K/I) \) in a natural way. The map \( f \mapsto \tilde{f} \) gives a \( \mathbb{C} \)-algebra isomorphism

\[ \mathbb{C}(B(k) \backslash G(k)/B(k)) \cong \mathbb{C}_c(I \backslash K/I). \]
Theorem (Iwahori)

\( \mathcal{H}_f \) is a free \( \mathbb{C} \)-algebra on generators \( T_{s_i} \) (indexed by the set \( S \) of simple reflections of \( W \)) subject to the following relations:

- \( T_{s_i}^2 = q + (q - 1) T_{s_i} \)
- \( T_{s_i} \cdot T_{s_j} \cdot T_{s_i} \ldots = T_{s_j} \cdot T_{s_i} \cdot T_{s_j} \ldots \quad (i \neq j) \)

with \( m_{ij} \) terms on both sides, with \( m_{ij} \geq 2 \), same as those occurring in the braid relations in the presentation of the Coxeter group \( W \) in terms of \( S \).
Theorem (Iwahori)

$\mathcal{H}_f$ is a free $\mathbb{C}$-algebra on generators $T_{s_i}$ (indexed by the set $S$ of simple reflections of $W$) subject to the following relations:

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- $T_{s_i} \cdot T_{s_j} \cdot T_{s_i} \ldots = T_{s_j} \cdot T_{s_i} \cdot T_{s_j} \ldots \ (i \neq j)$ with $m_{ij}$ terms on both sides, with $m_{ij} \geq 2$, same as those occurring in the braid relations in the presentation of the Coxeter group $W$ in terms of $S$.

$\mathcal{H}_f$ is very similar in structure to $\mathbb{C}[W]$. 
Definition

\[ \tilde{W}_a := N_{G(F)}(A(F))/A(O). \]
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Proposition (Iwahori and Matsumoto)

\[ G = \bigcup_{x \in \tilde{W}_a} lxl. \]
Extended affine Weyl group

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\[ \tilde{W}_a := N_{G(F)}(A(F))/A(\mathcal{O}). \]

Proposition (Iwahori and Matsumoto)

\[ G = \bigcup_{x \in \tilde{W}_a} lxl. \]

Thus the elements \( \{ T_x := 1_{lxl}, \ x \in \tilde{W}_a \} \) form a natural basis of \( \mathcal{H}_I \).
The extended affine Weyl group for $GL_n(F)$

Let $s_i \ (1 \leq i \leq n - 1)$ be the permutation matrix in $GL_n(F)$ corresponding to the transposition which interchanges $e_i$ and $e_{i+1}$. Further let

$$s_0 = \begin{pmatrix} 0 & 1 & & & \pi^{-1} \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ \pi & & & & 0 \end{pmatrix}, \quad t = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$
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Note that $ts_i t^{-1} = s_{i-1}$. 
\[ \tilde{W}_a = \langle s_0, \ldots, s_{n-1}, t \rangle \]

with the following relations:

- \( s_i^2 = 1, \ 0 \leq i \leq n - 1, \)
- \((s_is_j)^{m_{ij}} = 1\) where \( m_{i,i+1} = 3 \) and \( m_{i,j} = 2 \) if \(|i - j| \mod n > 1, \)
- \( ts_it^{-1} = s_{i-1}. \)
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with the following relations:

- \( s_i^2 = 1, \ 0 \leq i \leq n - 1 \),
- \( (s_i s_j)^{m_{ij}} = 1 \) where \( m_{i,i+1} = 3 \) and \( m_{i,j} = 2 \) if \( \mid i - j \mid \mod n > 1 \),
- \( ts_i t^{-1} = s_{i-1} \).

Thus \( \tilde{W}_a \) is almost like \( S_n \).
Let length function, $\ell$ is defined to be the minimum number of generators of the type $s_i$ (i.e. excluding any occurrences of the generator $t$). Note that $\ell$ restricted to $W$ is the usual length function.
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Notice that $s_1, \ldots, s_{n-1}$ generate $S_n$ while if we adjoin $s_0$ and $t$ we get all diagonal matrices, whose entries are integral powers of $\pi$. 
Let length function, \( \ell \) is defined to be the minimum number of generators of the type \( s_i \) (i.e. excluding any occurrences of the generator \( t \)). Note that \( \ell \) restricted to \( W \) is the usual length function.

Notice that \( s_1, \ldots, s_{n-1} \) generate \( S_n \) while if we adjoin \( s_0 \) and \( t \) we get all diagonal matrices, whose entries are integral powers of \( \pi \).

It can be easily checked that \( \tilde{W}_a \cong S_n \times \mathbb{Z}^n \) where \( \mathbb{Z}^n \) is embedded as the diagonal matrices with powers of \( \pi \).
The Iwahori-Matsumoto presentation for $G = \text{GL}_n(F)$

**Theorem**

$\mathcal{H}_i$ is an algebra generated by appropriately indexed $T_{s_i}$ and $T_t$ such that the following relations are satisfied:

- $T_{s_i} \cdot T_{s_i} = q + (q - 1) T_{s_i}$.
- $T_w \cdot T_{w'} = T_{ww'}$ if $\ell(w w') = \ell(w) + \ell(w')$.
- $T_{s_{i-1}} \cdot T_t = T_t \cdot T_{s_i}$.

(Here $i + 1$ is to be interpreted as 0 if $i = n - 1$.)
The co-character lattice

\[ X_*(A) := \text{Hom}(\mathbb{G}_m, A(F)) \]
\[ \pi^\mu := \mu(\pi), \quad \forall \mu \in X_*(A) \]
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**Lemma**

_The map \( \mu \mapsto \pi^\mu \) is an isomorphism_

\[ X_*(A) \cong A(F)/A(\mathcal{O}) \cong \mathbb{Z}^n. \]
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**Lemma**

The map \( \mu \mapsto \pi^\mu \) is an isomorphism

\[ X_*(A) \cong A(F)/A(\mathcal{O}) \cong \mathbb{Z}^n. \]

In case of \( \text{GL}_n(F) \),

\[ \mu \in X_*(A) \leftrightarrow \text{diag}(\pi^{t_1}, \ldots, \pi^{t_n}) \leftrightarrow (t_1, \ldots, t_n) \]

where \((t_1, \ldots, t_n) \in \mathbb{Z} \).
Definition

Call an element $\mu \in X_\ast(A)$ dominant if

$$\pi^\mu (I \cap N) \pi^{-\mu} \subset I \cap N.$$ 

Call $\mu$ antidominant if $-\mu$ is dominant.
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For example, in the case of $GL_n(F)$, this means that $\mu$ corresponds to $(t_1, \ldots, t_n)$ with $t_1 \geq t_2 \geq \ldots \geq t_n$. 
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It follows that $\pi^{-\mu} (I \cap \bar{N}) \pi^\mu \subset I \cap \bar{N}$.
Let $\mathcal{B} = \mathbb{C}[X_*(A)]$. Using the description of $X_*(A)$,

$$\mathcal{B} \cong \mathbb{C}[X_1, ..., X_n, X_1^{-1}, ..., X_n^{-1}].$$
The toric subalgebra

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For $\mu \in X_*(A)$ dominant define

$$\Theta_\mu = \delta^{1/2}(\pi^\mu) T_{\pi^\mu},$$

where $\delta$ is the modulus function of $B(F)$. 
The toric subalgebra

Let \( \mathcal{B} = \mathbb{C}[X_*(A)] \). Using the description of \( X_*(A) \),

\[
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For \( \mu \in X_*(A) \) dominant define

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\Theta_{\mu} = \delta^{1/2}(\pi^\mu) T_{\pi^\mu},
\]

where \( \delta \) is the modulus function of \( B(F) \).

Proposition

\( \Theta_{\mu} \) is invertible in \( \mathcal{H}_I \).
Let $\mu$ and $\eta$ are dominant. Then,

$$I_{\pi^\mu} I_{\pi^\eta} = I_{\pi^\mu(I \cap N)(I \cap A)(I \cap \bar{N})_{\pi^\eta}} I_{\in I_{\pi^\mu(I \cap N)_{\pi^\eta}(I \cap \bar{N})} I_{\in I_{\pi^\mu\pi^\eta}}}$$
Let $\mu$ and $\eta$ are dominant. Then,

\[ I_{\pi \mu} I_{\pi \eta} I = I_{\pi \mu} (I \cap N)(I \cap A)(I \cap \bar{N})_{\pi \eta} I \]
\[ \subset I (I \cap N_{\pi \mu} (I \cap A)_{\pi \eta} (I \cap \bar{N})_{\pi \eta} I \]
\[ \subset I_{\pi \mu} I_{\pi \eta} I \]

So we have, $I_{\pi \mu} I_{\pi \eta} I = I_{\pi \mu + \eta} I$, which implies

\[ \Theta_{\mu} \ast \Theta_{\eta} = \Theta_{\mu + \eta} \]
For $\mu \in X_*(A)$, write $\mu = \mu^+ - \mu^-$, where $\mu^+, \mu^-$ are dominant. So for an arbitrary $\mu$ define

$$\Theta_\mu = \Theta_{\mu^+} \ast (\Theta_{\mu^-})^{-1}.$$
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**Proposition**

The map $\Theta : B \to H_I$ is an isomorphism onto its image.

$\Theta(B)$ is an abelian subalgebra of $H_I$ and is known as the toric subalgebra.
The Bernstein relations

We have the following commutation relation.

**Proposition**

Let \( \mu \in X_*(A) \), \( s \in S \) and \( \alpha \) the simple root corresponding to \( s \). Then

\[
T_s \ast \Theta_{\mu} = \Theta_{s\mu} \ast T_s + (q - 1) \frac{\Theta_{\mu} - \Theta_{s\mu}}{1 - \Theta_{-\tilde{\alpha}}}.
\]

\( (s\mu = \mu - \langle \alpha, \mu \rangle \tilde{\alpha}) \)
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T_s * \Theta_\mu = \Theta_{s\mu} * T_s + (q - 1) \frac{\Theta_\mu - \Theta_{s\mu}}{1 - \Theta_{-\check{\alpha}}}.
\]

\((s\mu = \mu - \langle \alpha, \mu \rangle \check{\alpha})\)

\[
\frac{\Theta_\mu - \Theta_{s\mu}}{1 - \Theta_{-\check{\alpha}}} = \Theta_\mu + \Theta_{\mu - \check{\alpha}} + \ldots + \Theta_{s\mu + \check{\alpha}}
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Let $\mu \in \mathcal{X}_*(A)$, $s \in S$ and $\alpha$ the simple root corresponding to $s$. Then

$$T_s \ast \Theta_\mu = \Theta_{s\mu} \ast T_s + (q - 1) \frac{\Theta_\mu - \Theta_{s\mu}}{1 - \Theta_{-\check{\alpha}}}.$$  

($s\mu = \mu - \langle \alpha, \mu \rangle \check{\alpha}$)

$$\Theta_\mu - \Theta_{s\mu} \quad \frac{1}{1 - \Theta_{-\check{\alpha}}} = \Theta_\mu + \Theta_{\mu - \check{\alpha}} + ... + \Theta_{s\mu + \check{\alpha}}$$

**Proposition**

The elements $\Theta_\mu T_w \in \mathcal{H}_I$ form a basis over $\mathbb{C}$. 
The Bernstein presentation

Let $\mathcal{H}$ be the algebra with the generators

$$\{ T_w, w \in W, \Theta_\mu, \mu \in X_*(A) \}$$

such that the following relations are satisfied:

- $T_s \cdot T_s = q + (q - 1) T_s \ \forall s \in S$.  
- $T_w \cdot T_{w'} = T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$.  
- $\Theta_\mu \cdot \Theta_\eta = \Theta_{\mu+\eta}$.  
- For $s = s_\alpha \in S, \mu \in \mathcal{B}$ we have,

$$T_s \cdot \Theta_\mu = \Theta_{s_\mu} \cdot T_s + (q - 1) \frac{\Theta_\mu - \Theta_{s_\mu}}{1 - \Theta_{-\alpha}}.$$  

Then $\mathcal{H} \cong \mathcal{H}_I$.  

$W$ acts naturally on $B$. It follows easily from the commutation relation that

$$\Theta(B^W) = \text{Span} < \sum_{w \in W} \Theta_{w(\mu)} > \subset Z(H_I).$$
Description of the center

\( \mathcal{W} \) acts naturally on \( \mathcal{B} \). It follows easily from the commutation relation that

\[
\Theta(\mathcal{B}^W) = \text{Span} \left< \sum_{w \in \mathcal{W}} \Theta_w(\mu) \right> \subset Z(\mathcal{H}_I).
\]

Theorem (Bernstein)

\[
\Theta : \mathcal{B}^W \simrightarrow Z(\mathcal{H}_I).
\]
\( W \) acts naturally on \( B \). It follows easily from the commutation relation that

\[
\Theta(B^W) = \text{Span} < \sum_{w \in W} \Theta_{w(\mu)} > \subset Z(H_I).
\]

**Theorem (Bernstein)**

\[
\Theta : B^W \isomorph Z(H_I).
\]

It is easy to check that \( H_I \) is a f.g. module over \( B \) and \( B \) over \( B^W \). Thus \( (H_I) \) is f.g. over \( Z(H_I) \) and hence is noetherian.
Structure of $\mathcal{H}_I$ is governed by $\tilde{W}_a$.

$\mathcal{H}_I \cong \mathcal{H}_f \otimes_{\mathbb{C}} \mathcal{B}$ (as $\mathbb{C}$-vector spaces). ($\mathcal{B}$ abelian)

$T_s \cdot \Theta_\mu = \Theta_{s\mu} \cdot T_s + (q - 1) \frac{\Theta_\mu - \Theta_{s\mu}}{1 - \Theta_{-\tilde{\alpha}}}.$

$Z(\mathcal{H}_I) \cong \mathcal{B}^W.$
Thank you for your attention.