Structure of p-adic groups

Venketasubramanian. C. G.
Ben-Gurion University of the Negev

University of Copenhagen, 19 August 2013
Plan of this talk

- Notation, introduce main objects via examples in $GL_n$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
- Levi subgroups, Levi decomposition
- Three Facts about Lattices in $F^n$
- Bruhat, Iwasawa, Cartan, Iwahori decompositions
- Applications of each decomposition
Notation, introduce main objects via examples in $GL(n)$
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
- Levi subgroups, Levi decomposition
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
- Levi subgroups, Levi decomposition
- Three Facts about Lattices in $F^n$
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
- Levi subgroups, Levi decomposition
- Three Facts about Lattices in $F^n$
- Bruhat, Iwasawa, Cartan, Iwahori decompositions
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
- Levi subgroups, Levi decomposition
- Three Facts about Lattices in $F^n$
- Bruhat, Iwasawa, Cartan, Iwahori decompositions
- Applications of each decomposition
Plan of this talk

- Notation, introduce main objects via examples in $GL(n)$
- Torus, Borel subgroup, Weyl group
- Maximal compact subgroups, Parabolic subgroups
- Levi subgroups, Levi decomposition
- Three Facts about Lattices in $F^n$
- Bruhat, Iwasawa, Cartan, Iwahori decompositions
- Applications of each decomposition
Notation

- $F$: non-archimedean local field
- $\nu_F$: the discrete valuation on $F$
- $O_F$: ring of integers in $F$
- $P_F$: unique maximal ideal of $O_F$
- $\omega_F$: uniformizer of $F$

$P_F = \langle \omega_F \rangle$
Notation

- $F$ non-archimedean local field

Venketasubramanian. C. G. Ben-Gurion University of the Nege Structure of p-adic groups
- $F$ non-archimedean local field
- $\nu_F$ the discrete valuation on $F$
- $F$ non-archimedean local field
- $\nu_F$ the discrete valuation on $F$
- $O_F$ ring of integers in $F$
- \( F \) non-archimedean local field
- \( \nu_F \) the discrete valuation on \( F \)
- \( O_F \) ring of integers in \( F \)
- \( P_F \) unique maximal ideal of \( O_F \)
Notation

- $F$ non-archimedean local field
- $\nu_F$ the discrete valuation on $F$
- $O_F$ ring of integers in $F$
- $P_F$ unique maximal ideal of $O_F$
- $\omega_F$ uniformizer of $F$
Notation

- $F$ non-archimedean local field
- $\nu_F$ the discrete valuation on $F$
- $O_F$ ring of integers in $F$
- $P_F$ unique maximal ideal of $O_F$
- $\omega_F$ uniformizer of $F$
- $P_F = \langle \omega_F \rangle$
Notation

- $F$ non-archimedean local field
- $\nu_F$ the discrete valuation on $F$
- $O_F$ ring of integers in $F$
- $P_F$ unique maximal ideal of $O_F$
- $\omega_F$ uniformizer of $F$
- $P_F = \langle \omega_F \rangle$
Notation

- $F$ non-archimedean local field
- $\nu_F$ the discrete valuation on $F$
- $O_F$ ring of integers in $F$
- $P_F$ unique maximal ideal of $O_F$
- $\omega_F$ uniformizer of $F$
- $P_F = \langle \omega_F \rangle$
Key Players I

Maximal Torus; $\mathbf{T} = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$

Borel subgroup; $\mathbf{B} = \begin{pmatrix} a_1^* & \cdots & a_n^* \end{pmatrix} \subset \text{GL}(n)$

Weyl Group: $W(G, T) := \mathcal{N}_G(T)/T = \{ [g_{ij}] : \forall i, \exists \text{unique } j \text{ s.t. } g_{ij} \neq 0 \}$

$\mathcal{N}_G(T)/T \cong \text{S}_n$

Venketasubramanian. C. G. Ben-Gurion University of the Negev

Structure of p-adic groups
Key Players I

- Maximal Torus;

\[
\begin{pmatrix}
a_1 & \cdots & a_n \\
\end{pmatrix}
\]

Borel subgroup;

\[
\begin{pmatrix}
a_1^* & \cdots & a_n^* \\
\end{pmatrix}
\subset GL(n)
\]

Weyl Group:

\[W(G, T) := N_G(T) / T \cong \{ [g_{ij}] : \forall i, \exists \text{unique } j \text{ s.t } g_{ij} \neq 0 \}\]

Venketasubramanian. C. G. Ben-Gurion University of the Negev  Structure of p-adic groups
Key Players

- Maximal Torus; $T = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$
Key Players I

- Maximal Torus; $T = \begin{pmatrix} a_1 & \cdots \\ \vdots \\ a_n \end{pmatrix}$

- Borel subgroup;
Key Players I

- Maximal Torus; \( T = \begin{pmatrix} a_1 & \cdots & \cdots & a_n \\ & \ddots & \cdots & \end{pmatrix} \subset GL(n) \)

- Borel subgroup; \( B = \begin{pmatrix} a_1 & * & * \\ & \ddots & \cdots & * \\ & & a_n \end{pmatrix} \subset GL(n) \)
Key Players I

- Maximal Torus: \( T = \begin{pmatrix} a_1 & \cdots & \cdots \\ \cdots & & \cdots \\ \cdots & \cdots & a_n \end{pmatrix} \)

- Borel subgroup: \( B = \begin{pmatrix} a_1 & * & * \\ \cdots & & \cdots \\ & \cdots & * \end{pmatrix} \subset GL(n) \)

- Weyl Group: \( W(G, T) := N_G(T)/T \)

Venketasubramanian. C. G. Ben-Gurion University of the Negev Structure of p-adic groups
Maximal Torus; $T = \begin{pmatrix} a_1 & \cdots & \cdots & a_n \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_1 & \cdots & \cdots & a_n \end{pmatrix}$

Borel subgroup; $B = \begin{pmatrix} a_1 & * & * \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & \ddots \\ a_1 & * & * \end{pmatrix} \subseteq GL(n)$

Weyl Group: $W(G, T) := N_G(T)/T$

$N_G(T) = \{ [g_{ij}] : \forall i, \exists j \text{ unique s.t } g_{ij} \neq 0 \}$
Key Players I

- Maximal Torus; \( T = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \)

- Borel subgroup; \( B = \begin{pmatrix} a_1 & * & * \\ & \ddots & * \\ & & a_n \end{pmatrix} \subset GL(n) \)

- Weyl Group: \( W(G, T) := \frac{N_G(T)}{T} \)

- \( N_G(T) = \{[g_{ij}] : \forall i, \exists \text{ unique } j \text{ s.t } g_{ij} \neq 0\} \)

- \( N_G(T)/T \cong S_n \)
Maximal Torus; $T = \begin{pmatrix} a_1 & \cdot & \cdot & \cdot & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{pmatrix}$

Borel subgroup; $B = \begin{pmatrix} a_1 & * & * \\ \cdot & \cdot & * \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_n \end{pmatrix} \subset GL(n)$

Weyl Group: $W(G, T) := N_G(T)/T$

$N_G(T) = \{ [g_{ij}] : \forall i, \exists \text{ unique } j \text{ s.t } g_{ij} \neq 0 \}$

$N_G(T)/T \simeq S_n$
$B$ subgroup of upper triangular matrices in $GL(n, F)$
Bruhat Decomposition

- $B$ subgroup of upper triangular matrices in $GL(n, F)$
- $T \subset B$ maximal torus; group of diagonal matrices
Bruhat Decomposition

- $B$ subgroup of upper triangular matrices in $GL(n, F)$
- $T \subset B$ maximal torus; group of diagonal matrices
- $g \in GL(n, F)$: $i < j$ adding $aR_i$ to $R_j$ and $i > j$ adding $aC_i$ to $C_j$
Bruhat Decomposition

- $B$ subgroup of upper triangular matrices in $GL(n, F)$
- $T \subset B$ maximal torus; group of diagonal matrices
- $g \in GL(n, F)$: $i < j$ adding $aR_i$ to $R_j$ and $i > j$ adding $aC_i$ to $C_j$
- $b_1, b_2 \in B$ such that $b_1gb_2 = dw$ where $d \in T, w \in W$
Bruhat Decomposition

- $B$ subgroup of upper triangular matrices in $GL(n, F)$
- $T \subset B$ maximal torus; group of diagonal matrices
- $g \in GL(n, F)$: $i < j$ adding $aR_i$ to $R_j$ and $i > j$ adding $aC_i$ to $C_j$
- $b_1, b_2 \in B$ such that $b_1gb_2 = dw$ where $d \in T$, $w \in W$

Theorem

We have the Bruhat decomposition

$$GL(n) = \coprod BwB$$
Bruhat Decomposition

- $B$ subgroup of upper triangular matrices in $GL(n, F)$
- $T \subset B$ maximal torus; group of diagonal matrices
- $g \in GL(n, F)$: $i < j$ adding $aR_i$ to $R_j$ and $i > j$ adding $aC_i$ to $C_j$
- $b_1, b_2 \in B$ such that $b_1gb_2 = dw$ where $d \in T, w \in W$

**Theorem**

We have the Bruhat decomposition

$$GL(n) = \bigsqcup BwB$$
An application of Bruhat Decomposition

$G = GL(2)$

A filtration for $\text{ind}_G^B (\chi)$

$B \curvearrowright B \setminus G$: There are two orbits.

$B$ closed and $BwB$ open

Use: $0 \rightarrow S(Y) \rightarrow S(X) \rightarrow S(Z) \rightarrow 0$

Venketasubramanian. C. G. Ben-Gurion University of the Negev Structure of p-adic groups
An application of Bruhat Decomposition

$G = GL(2)$
An application of Bruhat Decomposition

- $G = GL(2)$
- A filtration for $\text{ind}_B^G(\chi)$
An application of Bruhat Decomposition

- $G = GL(2)$
- A filtration for $\text{ind}_B^G(\chi)$
- $B \curvearrowright B \backslash G$: There are two orbits.
An application of Bruhat Decomposition

- $G = GL(2)$
- A filtration for $\text{ind}^G_B(\chi)$
- $B \lhd B \backslash G$: There are two orbits.
- $B$ closed and $BwB$ open
An application of Bruhat Decomposition

- $G = GL(2)$
- A filtration for $\text{ind}_{B}^{G}(\chi)$
- $B \actson B \backslash G$: There are two orbits.
- $B$ closed and $BwB$ open
- Use: $0 \rightarrow S(Y) \rightarrow S(X) \rightarrow S(Z) \rightarrow 0$
An application of Bruhat Decomposition

- $G = GL(2)$
- A filtration for $\text{ind}_B^G(\chi)$
- $B \lhd B \backslash G$: There are two orbits.
- $B$ closed and $BwB$ open
- Use: $0 \to S(Y) \to S(X) \to S(Z) \to 0$
Parabolic subgroups, Unipotent Radical, Levi subgroups

\[ n = n_1 + \cdots + n_k \]

\[ P_{n_1}, \ldots, n_k = \begin{pmatrix} 
GL(n_1) & * & \cdots & * \\
& GL(n_k) & & 
\end{pmatrix} \]

\[ N_{n_1}, \ldots, n_k = \begin{pmatrix} 
I_{n_1} & * & \cdots & * \\
& I_{n_k} & & 
\end{pmatrix} \]

\[ M_{n_1}, \ldots, n_k = \begin{pmatrix} 
GL(n_1) & \cdots & GL(n_k) 
\end{pmatrix} \]
Key Players II

- Parabolic subgroups, Unipotent Radical, Levi subgroups
Key Players II

- Parabolic subgroups, Unipotent Radical, Levi subgroups
- \( n = n_1 + \cdots + n_k \)
Key Players II

- Parabolic subgroups, Unipotent Radical, Levi subgroups
- \( n = n_1 + \cdots + n_k \)

\[
P_{n_1, \ldots, n_k} = \begin{pmatrix}
GL(n_1) & \ast & \ast \\
\ast & \ddots & \ast \\
\ast & \ast & GL(n_k)
\end{pmatrix}
\]
Key Players II

- Parabolic subgroups, Unipotent Radical, Levi subgroups
- $n = n_1 + \cdots + n_k$

$$
P_{n_1, \ldots, n_k} = \begin{pmatrix}
GL(n_1) & * & * \\
& \ddots & * \\
& & GL(n_k)
\end{pmatrix}
$$

$$
N_{n_1, \ldots, n_k} = \begin{pmatrix}
I_{n_1} & * & * \\
& \ddots & * \\
& & I_{n_k}
\end{pmatrix}
$$
Key Players II

- Parabolic subgroups, Unipotent Radical, Levi subgroups
- \( n = n_1 + \cdots + n_k \)

\[
P_{n_1, \ldots, n_k} = \begin{pmatrix}
\text{GL}(n_1) & * & * \\
& \ddots & * \\
& & \text{GL}(n_k)
\end{pmatrix}
\]

\[
N_{n_1, \ldots, n_k} = \begin{pmatrix}
I_{n_1} & * & * \\
& \ddots & * \\
& & I_{n_k}
\end{pmatrix}
\]

\[
M_{n_1, \ldots, n_k} = \begin{pmatrix}
\text{GL}(n_1) \\
& \ddots \\
& & \text{GL}(n_k)
\end{pmatrix}
\]
Parabolic subgroups, Unipotent Radical, Levi subgroups

\[ n = n_1 + \cdots + n_k \]

\[ P_{n_1, \ldots, n_k} = \begin{pmatrix}
GL(n_1) & * & * \\
& \ddots & * \\
& & GL(n_k)
\end{pmatrix} \]

\[ N_{n_1, \ldots, n_k} = \begin{pmatrix}
I_{n_1} & * & * \\
& \ddots & * \\
& & I_{n_k}
\end{pmatrix} \]

\[ M_{n_1, \ldots, n_k} = \begin{pmatrix}
GL(n_1) \\
& \ddots \\
& & GL(n_k)
\end{pmatrix} \]
Levi decomposition

\[ N ⊴ P \quad \text{and} \quad N \setminus P \cong M_{n_1, \ldots, n_k} = M_{n_1, \ldots, n_k}^\times \]

\[ P_1, 1 = (GL(1) \times GL(1)) = (GL(1) \ 0 \ GL(1)) (1 \times 1) \]

\[ P_1, 2 = (GL(1) \times GL(2)) = (GL(1) \ 0 \ GL(2)) (1 \times \text{Id}(2)) \]
Levi decomposition
Levi decomposition

$N \trianglelefteq P$ and $N \setminus P \simeq M$
Levi decomposition

\[ N \trianglelefteq P \text{ and } N \backslash P \simeq M \]

\[ P_{n_1,\ldots,n_k} = M_{n_1,\ldots,n_k}N_{n_1,\ldots,n_k} \]
Levi decomposition

\[ N \trianglelefteq P \text{ and } N \backslash P \simeq M \]

\[ P_{n_1, \ldots, n_k} = M_{n_1, \ldots, n_k} N_{n_1, \ldots, n_k} \]

\[ P_{1,1} = \begin{pmatrix} \text{GL}(1) & * \\ \text{GL}(1) & \end{pmatrix} = \begin{pmatrix} \text{GL}(1) & 0 \\ \text{GL}(1) & \end{pmatrix} \begin{pmatrix} 1 & * \\ 1 & \end{pmatrix} \]

\[ P_{1,2} = \begin{pmatrix} \text{GL}(1) & * \\ \text{GL}(2) & \end{pmatrix} = \begin{pmatrix} \text{GL}(1) & 0 \\ \text{GL}(2) & \end{pmatrix} \begin{pmatrix} 1 & * \\ \text{Id}(2) & \end{pmatrix} \]
Recall:

**Theorem**

Let $R$ be a principal ideal domain, let $M$ be a free $R$-module of rank $n$, and let $N$ be a submodule of $M$ which is also free of rank $k$. Then there exists an $R$-basis $m_1, \cdots, m_n$ of $M$ and nonzero elements $r_1, \cdots, r_k$ of $R$ such that each $r_{i+1}$ divides $r_i$ \((i = 1, \cdots, k - 1)\) and $r_1 m_1, \cdots, r_k m_k$ is an $R$-basis of $N$. 
A lattice $L$ is an $O_F$-submodule of $F^n$ of rank $n$. 
Lattices-I

- A lattice $L$ is an $O_F$-submodule of $F^n$ of rank $n$.
- $\mathcal{L} :=$ the set of all lattices
A lattice $L$ is an $O_F$-submodule of $F^n$ of rank $n$.

$L :=$ the set of all lattices

**Proposition**

Given $L_1, L_2 \in \mathcal{L}$ there exists an $O_F$-basis $v_1, ..., v_n$ of $L_1$ and integers $m_1, ..., m_n$ such that $\omega^{m_1}v_1, ..., \omega^{m_n}v_n$ is an $O_F$ basis of $L_2$. 

Venketasubramanian, C. G. Ben-Gurion University of the Negev

Structure of p-adic groups
A lattice $L$ is an $O_F$-submodule of $F^n$ of rank $n$.

$L :=$ the set of all lattices

**Proposition**

Given $L_1, L_2 ∈ ℒ$ there exists an $O_F$-basis $v_1, ..., v_n$ of $L_1$ and integers $m_1, ..., m_n$ such that $ω^{m_1}v_1, ..., ω^{m_n}v_n$ is an $O_F$ basis of $L_2$.

Proof.
A lattice \( L \) is an \( O_F \)-submodule of \( F^n \) of rank \( n \).

\[ \mathcal{L} := \text{the set of all lattices} \]

**Proposition**

Given \( L_1, L_2 \in \mathcal{L} \) there exists an \( O_F \)-basis \( v_1, \ldots, v_n \) of \( L_1 \) and integers \( m_1, \ldots, m_n \) such that \( \omega^{m_1} v_1, \ldots, \omega^{m_n} v_n \) is an \( O_F \) basis of \( L_2 \).

**Proof.**

\[ r \in \mathbb{Z} \text{ with } \omega^r L_2 \subset L_1 \]
A lattice $L$ is an $O_F$-submodule of $F^n$ of rank $n$.

$\mathcal{L} :=$ the set of all lattices

**Proposition**

Given $L_1, L_2 \in \mathcal{L}$ there exists an $O_F$-basis $v_1, \ldots, v_n$ of $L_1$ and integers $m_1, \ldots, m_n$ such that $\omega^{m_1} v_1, \ldots, \omega^{m_n} v_n$ is an $O_F$ basis of $L_2$.

**Proof.**

- $r \in \mathbb{Z}$ with $\omega^r L_2 \subset L_1$
- $\exists$ basis $v_1, \ldots, v_n$ of $L_1$ and $0 \neq a_i \in O_F$ s.t. $a_1 v_1, \ldots, a_n v_n$ is a basis of $\omega^r L_2$
A lattice $L$ is an $OF$-submodule of $F^n$ of rank $n$.

$\mathcal{L} :=$ the set of all lattices

**Proposition**

Given $L_1, L_2 \in \mathcal{L}$ there exists an $OF$-basis $v_1, \ldots, v_n$ of $L_1$ and integers $m_1, \ldots, m_n$ such that $\omega^{m_1}v_1, \ldots, \omega^{m_n}v_n$ is an $OF$ basis of $L_2$.

**Proof.**

- $r \in \mathbb{Z}$ with $\omega^r L_2 \subset L_1$
- $\exists$ basis $v_1, \ldots, v_n$ of $L_1$ and $0 \neq a_i \in OF$ s.t. $a_1 v_1, \ldots, a_n v_n$ is a basis of $\omega^r L_2$
- $m_i = \nu F(a_i) - r$. $\square$
A lattice $L$ is an $O_F$-submodule of $F^n$ of rank $n$.  
\[ \mathcal{L} := \text{the set of all lattices} \]

**Proposition**

*Given $L_1, L_2 \in \mathcal{L}$ there exists an $O_F$-basis $v_1, \ldots, v_n$ of $L_1$ and integers $m_1, \ldots, m_n$ such that $\omega^{m_1}v_1, \ldots, \omega^{m_n}v_n$ is an $O_F$ basis of $L_2$.***

**Proof.**

\[ r \in \mathbb{Z} \text{ with } \omega^r L_2 \subset L_1 \]

\[ \exists \text{ basis } v_1, \ldots, v_n \text{ of } L_1 \text{ and } 0 \neq a_i \in O_F \text{ s.t. } a_1 v_1, \ldots, a_n v_n \]

is a basis of $\omega^r L_2$

\[ m_i = \nu_F(a_i) - r. \square \]
\begin{itemize}
  \item $L \in \mathcal{L}$; compact and open $O_F$-submodule of $F^n$.
\end{itemize}
\begin{itemize}
  \item $L \in \mathcal{L}$; compact and open $O_F$-submodule of $F^n$.
  \item $L_1, L_2 \in \mathcal{L}$; $L_1 + L_2 \in \mathcal{L}$.
\end{itemize}
• $L \in \mathcal{L}$; compact and open $O_F$-submodule of $F^n$.
• $L_1, L_2 \in \mathcal{L}$; $L_1 + L_2 \in \mathcal{L}$.
• $GL(n)$ acts transitively on $\mathcal{L}$ by $g.L = g(L)$
L \in \mathcal{L}; \text{ compact and open } O_F\text{-submodule of } F^n.

L_1, L_2 \in \mathcal{L}; L_1 + L_2 \in \mathcal{L}.

GL(n) \text{ acts transitively on } \mathcal{L} \text{ by } g.L = g(L)

L_0 = O^n_F \text{ then } \text{Stab}_{GL(n,F)}(L_0) = GL(n, O_F)
- $L \in \mathcal{L}$; compact and open $O_F$-submodule of $F^n$.
- $L_1, L_2 \in \mathcal{L}$; $L_1 + L_2 \in \mathcal{L}$.
- $GL(n)$ acts transitively on $\mathcal{L}$ by $g.L = g(L)$
- $L_0 = O^n_F$ then $\text{Stab}_{GL(n,F)}(L_0) = GL(n, O_F)$
- $K_0 := GL(n, O_F)$
\begin{itemize}
  \item $L \in \mathcal{L}$; compact and open $O_F$-submodule of $F^n$.
  \item $L_1, L_2 \in \mathcal{L}$; $L_1 + L_2 \in \mathcal{L}$.
  \item $GL(n)$ acts transitively on $\mathcal{L}$ by $g.L = g(L)$
  \item $L_0 = O_F^n$ then $\text{Stab}_{GL(n,F)}(L_0) = GL(n, O_F)$
  \item $K_0 := GL(n, O_F)$
\end{itemize}
Proposition

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$. 

Proof.

$L := \sum_{g \in S} gL_0$ is a lattice

simple calculation:

$K \subset Stab_{GL(n, F)}(L)$

$K$ is maximal;

$K = Stab_{GL(n, F)}(L)$

$K$ is conjugate to $K_0$.

□
Proposition

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$.

Proof.
Proposition

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$.

Proof.

- $K$ compact open, $S := K/(K \cap GL(n, O_F))$ is finite
Maximal compact subgroups

Proposition

All maximal compact subgroups of \( GL(n, F) \) are conjugate to \( K_0 \).

Proof.

- \( K \) compact open, \( S := K/(K \cap GL(n, O_F)) \) is finite
- \( L := \sum_{g \in S} gL_0 \) is a lattice
Maximal compact subgroups

Proposition

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$.

Proof.

- $K$ compact open, $S := K/(K \cap GL(n, O_F))$ is finite
- $L := \sum_{g \in S} gL_0$ is a lattice
- simple calculation: $K \subset Stab_{GL(n,F)}(L)$
**Proposition**

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$.

**Proof.**

- $K$ compact open, $S := K / (K \cap GL(n, O_F))$ is finite
- $L := \sum_{g \in S} gL_0$ is a lattice
- simple calculation: $K \subset Stab_{GL(n, F)}(L)$
- $K$ is maximal; $K = Stab_{GL(n, F)}(L)$
Proposition

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$.

Proof.

- $K$ compact open, $S := K/(K \cap GL(n, O_F))$ is finite
- $L := \sum_{g \in S} g L_0$ is a lattice
- simple calculation: $K \subset Stab_{GL(n, F)}(L)$
- $K$ is maximal; $K = Stab_{GL(n, F)}(L)$
- $K$ is conjugate to $K_0$. \qed
Maximal compact subgroups

Proposition

All maximal compact subgroups of $GL(n, F)$ are conjugate to $K_0$.

Proof.

- $K$ compact open, $S := K/(K \cap GL(n, O_F))$ is finite
- $L := \sum_{g \in S} gL_0$ is a lattice
- Simple calculation: $K \subset \text{Stab}_{GL(n, F)}(L)$
- $K$ is maximal; $K = \text{Stab}_{GL(n, F)}(L)$
- $K$ is conjugate to $K_0$. $\square$
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, ..., e_n$ be the standard basis for $F^n$. There exists an $O_{F^n}$-basis for $L$ given by

\[ v_1 = b_{11} e_1 + \cdots + b_{1n} e_n \]
\[ v_2 = b_{21} e_1 + b_{22} e_2 + \cdots + b_{2n} e_n \]
\[ \vdots \]
\[ v_n = b_{n1} e_1 + b_{n2} e_2 + \cdots + b_{nn} e_n \]

In other words, $\exists b \in \mathbb{B}$ s.t $b \cdot L = 0$.

Proof. An easy induction argument.
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, ..., e_n$ be the standard basis for $F^n$. There exists an $O_F$-basis for $L$ given by

$$
v_1 = b_{11}e_1 + b_{12}e_2 + \cdots + b_{1n}e_n
$$

$$
v_2 = b_{21}e_1 + b_{22}e_2 + \cdots + b_{2n}e_n
$$

... 

$$
v_n = b_{1n}e_1 + b_{2n}e_2 + \cdots + b_{nn}e_n
$$

In other words, \( \exists b \in B \text{ s.t. } b \cdot L = 0 \).

Proof. An easy induction argument.
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, ..., e_n$ be the standard basis for $F^n$. There exists an $O_F$-basis for $L$ given by

$$v_1 = b_{11}e_1$$

$$v_2 = b_{12}e_1 + b_{22}e_2$$

$$...$$

$$v_n = b_{1n}e_1 + b_{2n}e_2 + ... + b_{nn}e_n$$

In other words, $\exists b \in B$ s.t $b.L = L$

Proof. An easy induction argument.
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, ..., e_n$ be the standard basis for $F^n$. There exists an $O_F$-basis for $L$ given by

$$v_1 = b_{11}e_1$$
$$v_2 = b_{12}e_1 + b_{22}e_2$$

In other words, $\exists b \in B$ s.t $bL = L$

Proof. An easy induction argument.
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, ..., e_n$ be the standard basis for $F^n$. There exists an $O_F$-basis for $L$ given by

$$v_1 = b_{11}e_1$$
$$v_2 = b_{12}e_1 + b_{22}e_2$$
$$\vdots$$
$$v_n = b_{1n}e_1 + b_{2n}e_2 + \cdots + b_{nn}e_n$$

In other words, there exists a $b \in B$ such that $b.L = L$.

Proof. An easy induction argument.
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, ..., e_n$ be the standard basis for $F^n$. There exists an $O_F$-basis for $L$ given by

\begin{align*}
  v_1 &= b_{11}e_1 \\
  v_2 &= b_{12}e_1 + b_{22}e_2 \\
  &\vdots \\
  v_n &= b_{1n}e_1 + b_{2n}e_2 + \cdots + b_{nn}e_n
\end{align*}

In other words, $\exists \, b \in B \text{ s.t. } b.L_0 = L$
Lemma

Let $L$ be a lattice in $F^n$. Let $e_1, \ldots, e_n$ be the standard basis for $F^n$. There exists an $O_{F}$-basis for $L$ given by

$$v_1 = b_{11}e_1$$
$$v_2 = b_{12}e_1 + b_{22}e_2$$
$$\vdots$$
$$v_n = b_{1n}e_1 + b_{2n}e_2 + \cdots + b_{nn}e_n$$

In other words, $\exists b \in B$ s.t $b.L_0 = L$

Proof. An easy induction argument.
Iwasawa Decomposition

Theorem

Let $G = \text{GL}(n)$ and $K_0 = \text{GL}(n, \mathbb{O}_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition $G = BK_0$.

Proof. $g \in \text{GL}(n, F)$ and $L_0 = \text{On}_F L = gL_0$ by Lemma $L = bL_0$ with $b \in B$. $b^{-1}g$ stabilizes $L_0$ and hence is in $K_0$. □

More generally, if $P$ is any parabolic subgroup and $K$ is any maximal compact subgroup of $\text{GL}(n)$ then $\text{GL}(n) = PK$. 

Venkatesubramanian. C. G. Ben-Gurion University of the Negev

Structure of p-adic groups
Iwasawa Decomposition

**Theorem**

Let $G = GL(n)$ and $K_0 = GL(n, O_F)$. 

Proof. 

Let $g \in GL(n, F)$ and $L_0 = O_n$. Let $L = gL_0$ with $b \in B$. $b^{-1}g$ stabilizes $L_0$ and hence is in $K_0$. 

$\blacksquare$

More generally, if $P$ is any parabolic subgroup and $K$ is any maximal compact subgroup of $GL(n)$ then $GL(n) = PK$.
Let $G = \text{GL}(n)$ and $K_0 = \text{GL}(n, O_F)$. Let $B$ denote the group of upper triangular matrices.

**Proof.**

Let $g \in \text{GL}(n, F)$ and $L_0 = O_n$. Then $gL_0 = bL_0$ with $b \in B$. $b^{-1}g$ stabilizes $L_0$ and hence is in $K_0$.

□

More generally, if $P$ is any parabolic subgroup and $K$ is any maximal compact subgroup of $\text{GL}(n)$ then $\text{GL}(n) = PK$. 

Venketasubramanian. C. G. Ben-Gurion University of the Negev  Structure of p-adic groups
Iwasawa Decomposition

Theorem

Let $G = \text{GL}(n)$ and $K_0 = \text{GL}(n, O_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

\[ G = BK_0 \]
Iwasawa Decomposition

**Theorem**

Let $G = GL(n)$ and $K_0 = GL(n, O_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

**Proof.**
Iwasawa Decomposition

Theorem

Let $G = GL(n)$ and $K_0 = GL(n, O_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

Proof.

- $g \in GL(n, F)$ and $L_0 = O^n_F$
Iwasawa Decomposition

**Theorem**

Let $G = \text{GL}(n)$ and $K_0 = \text{GL}(n, O_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

**Proof.**

- $g \in \text{GL}(n, F)$ and $L_0 = O^n_F$
- $L = gL_0$
Let $G = GL(n)$ and $K_0 = GL(n, O_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

Proof.

- $g \in GL(n, F)$ and $L_0 = O_F^n$
- $L = gL_0$
- By Lemma $L = bL_0$ with $b \in B$
Iwasawa Decomposition

Theorem

Let $G = GL(n)$ and $K_0 = GL(n, O_F)$. Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

Proof.

- $g \in GL(n, F)$ and $L_0 = O^n_F$
- $L = g L_0$
- By Lemma $L = bL_0$ with $b \in B$
- $b^{-1}g$ stabilizes $L_0$ and hence is in $K_0$. □
Iwasawa Decomposition

Theorem

Let $G = GL(n)$ and $K_0 = GL(n, O_F).$ Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

Proof.

- $g \in GL(n, F)$ and $L_0 = O^n_F$
- $L = gL_0$
- By Lemma $L = bL_0$ with $b \in B$
- $b^{-1}g$ stabilizes $L_0$ and hence is in $K_0$. □

More generally, if $P$ is any parabolic subgroup and $K$ is any maximal compact subgroup of $GL(n)$ then
Theorem

Let $G = GL(n)$ and $K_0 = GL(n, O_F).$ Let $B$ denote the group of upper triangular matrices. Then we have the decomposition

$$G = BK_0$$

Proof.

- $g \in GL(n, F)$ and $L_0 = O_F^n$
- $L = g L_0$
- By Lemma $L = b L_0$ with $b \in B$
- $b^{-1} g$ stabilizes $L_0$ and hence is in $K_0$. □

More generally, if $P$ is any parabolic subgroup and $K$ is any maximal compact subgroup of $GL(n)$ then

$$GL(n) = PK$$
An application of Iwasawa Decomposition

If \( \pi \) is finitely generated then \( r_M, \text{GL}(n)(\pi) \) is finitely generated

Proof. \( V \) be generated by a finite set \( S \)

\[ V = \text{Span}\{ \pi(G)S \} \]

Iwasawa Decomposition \( \Rightarrow V = \text{Span}\{ \pi(P)\pi(K)S \} \)

\( \pi \) is smooth \( \Rightarrow \pi(K)S \) is finite

\( V \) is finitely generated as \( P \)-module

\( V_N \) is finitely generated as an \( M \)-module.

\( \square \)
An application of Iwasawa Decomposition

If $\pi$ is finitely generated then $r_{M, GL(n)}(\pi)$ is finitely generated.
If \( \pi \) is finitely generated then \( r_{M,GL(n)}(\pi) \) is finitely generated.

Proof.

\[ V = \text{Span}\{ \pi(G)S \} \]
\[ \Rightarrow V = \text{Span}\{ \pi(P)\pi(K)S \} \]

\[ \pi \text{ is smooth} \Rightarrow \pi(K)S \text{ is finite} \]

\( V \) is finitely generated as a \( P \)-module.

\( V_N \) is finitely generated as an \( M \)-module.

\[ \square \]
If \( \pi \) is finitely generated then \( r_{M, GL(n)}(\pi) \) is finitely generated.

Proof.

- \( V \) be generated by a finite set \( S \)
If \( \pi \) is finitely generated then \( r_{M,\text{GL}(n)}(\pi) \) is finitely generated

Proof.

- \( V \) be generated by a finite set \( S \)
- \( V = \text{Span}\{\pi(G)S\} \)
If $\pi$ is finitely generated then $r_{\mathcal{M}, \text{GL}(n)}(\pi)$ is finitely generated

Proof.

- $V$ be generated by a finite set $S$
- $V = \text{Span}\{\pi(G)S\}$
- Iwasawa Decomposition $\implies V = \text{Span}\{\pi(P)\pi(K)S\}$
If $\pi$ is finitely generated then $r_{M,GL(n)}(\pi)$ is finitely generated.

Proof.

- $V$ be generated by a finite set $S$
- $V = \text{Span}\{\pi(G)S\}$
- Iwasawa Decomposition $\Rightarrow V = \text{Span}\{\pi(P)\pi(K)S\}$
- $\pi$ is smooth $\Rightarrow \pi(K)S$ is finite
If \( \pi \) is finitely generated then \( r_{M, GL(n)}(\pi) \) is finitely generated.

**Proof.**

- \( V \) be generated by a finite set \( S \)
- \( V = \text{Span}\{\pi(G)S\} \)
- Iwasawa Decomposition \( \implies \) \( V = \text{Span}\{\pi(P)\pi(K)S\} \)
- \( \pi \) is smooth \( \implies \) \( \pi(K)S \) is finite
- \( V \) is finitely generated as \( P \)-module
If $\pi$ is finitely generated then $r_{M, GL(n)}(\pi)$ is finitely generated.

Proof.

- $V$ be generated by a finite set $S$
- $V = \text{Span}\{\pi(G)S\}$
- Iwasawa Decomposition $\implies V = \text{Span}\{\pi(P)\pi(K)S\}$
- $\pi$ is smooth $\implies \pi(K)S$ is finite
- $V$ is finitely generated as $P$-module
- $V_N$ is finitely generated as an $M$-module. $\square$
An application of Iwasawa Decomposition

If \( \pi \) is finitely generated then \( r_{M,\text{GL}(n)}(\pi) \) is finitely generated.

Proof.

1. \( V \) be generated by a finite set \( S \)
2. \( V = \text{Span}\{\pi(G)S\} \)
3. Iwasawa Decomposition \( \implies \) \( V = \text{Span}\{\pi(P)\pi(K)S\} \)
4. \( \pi \) is smooth \( \implies \) \( \pi(K)S \) is finite
5. \( V \) is finitely generated as \( P \)-module
6. \( V_N \) is finitely generated as an \( M \)-module. \( \square \)
Cartan Decomposition

Theorem

Let $K_0 = \text{GL}(n, \mathbb{O}_{\mathbb{F}})$. Let

$$A^+ = \left\{ \text{diag}(\omega_{m_1}, \ldots, \omega_{m_n}) : m_1, \ldots, m_n \in \mathbb{Z}, m_1 \leq \ldots \leq m_n \right\}.$$

Then, we have the decomposition

$$\text{GL}(n, \mathbb{F}) = K_0 A^+ K_0.$$

Proof.

$g \in \text{GL}(n, \mathbb{F})$; put $L = gL_0$ where $L_0 = \mathbb{O}_n$. There exists a basis $v_1, \ldots, v_n$ of $L_0$ s.t.

$$\omega_{m_1}v_1, \ldots, \omega_{m_n}v_n$$

is a basis of $L$. Let

$$k = [v_1, \ldots, v_n] \in K_0$$

and

$$a = \text{diag}(\omega_{m_1}, \ldots, \omega_{m_n}).$$

Simple calculation:

$$k^{-1}ak \in K_0.$$ 

$\square$

Venketasubramanian. C. G. Ben-Gurion University of the Negev

Structure of p-adic groups
Theorem

Let $K_0 = GL(n, O_F)$. 

Proof. 

$g \in GL(n, F)$; put $L_0 = gL_0$ where $L_0 = O_{n, F}$.

$\exists$ basis $v_1, \ldots, v_n$ of $L_0$ s.t $\omega_{m_1}v_1, \ldots, \omega_{m_n}v_n$ is a basis of $L_0$.

$k := [v_1, \ldots, v_n] \in K_0$ and $a = diag(\omega_{m_1}, \ldots, \omega_{m_n})$.

Simple calculation: $k^{-1}ak \in K_0$.

$\square$
Theorem

Let $K_0 = GL(n, O_F)$. Let

$A^+ = \{\text{diag}(\omega^{m_1}, \ldots, \omega^{m_n}) : m_1, \ldots, m_n \in \mathbb{Z}, m_1 \leq \ldots \leq m_n\}$. 

Proof.

$g \in GL(n, F)$; put $L = gL_0$ where $L_0 = O_n_F$. ∃ basis $v_1, \ldots, v_n$ of $L_0$ s.t $\omega^{m_1}v_1, \ldots, \omega^{m_n}v_n$ is a basis of $L$. 

$k := [v_1, \ldots, v_n] \in K_0$ and $a = \text{diag}(\omega^{m_1}, \ldots, \omega^{m_n})$. 

Simple calculation: 

$k^{-1}akL_0 = Lg^{-1}k^{-1}ak \in K_0$.

□
Theorem

Let $K_0 = GL(n, O_F)$. Let

$$A^+ = \{ \text{diag}(\omega^{m_1}, ..., \omega^{m_n}) : m_1, ..., m_n \in \mathbb{Z}, m_1 \leq ... \leq m_n \}.$$ 

Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$
Cartan Decomposition

Theorem

Let $K_0 = GL(n, O_F)$. Let

$$A^+ = \{ \text{diag}(\omega^{m_1}, \ldots, \omega^{m_n}) : m_1, \ldots, m_n \in \mathbb{Z}, m_1 \leq \ldots \leq m_n \}.$$  

Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$  

Proof.
Let $K_0 = GL(n, O_F)$. Let $A^+ = \{ \text{diag}(\omega^{m_1}, \ldots, \omega^{m_n}) : m_1, \ldots, m_n \in \mathbb{Z}, m_1 \leq \ldots \leq m_n \}$. Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$ 

Proof. $g \in GL(n, F)$; put $L = gL_0$ where $L_0 = O_F^n$. 
Theorem

Let \( K_0 = \text{GL}(n, O_F) \). Let \( A^+ = \{ \text{diag}(\omega^{m_1}, \ldots, \omega^{m_n}) : m_1, \ldots, m_n \in \mathbb{Z}, m_1 \leq \ldots \leq m_n \} \). Then, we have the decomposition

\[ \text{GL}(n, F) = K_0 A^+ K_0. \]

Proof. \( g \in \text{GL}(n, F) \); put \( L = gL_0 \) where \( L_0 = O_F^n \).

\( \exists \) basis \( v_1, \ldots, v_n \) of \( L_0 \) s.t. \( \omega^{m_1} v_1, \ldots, \omega^{m_n} v_n \) is a basis of \( L \).
Cartan Decomposition

Theorem

Let $K_0 = GL(n, O_F)$. Let

$A^+ = \{ \text{diag}(\omega^{m_1}, ..., \omega^{m_n}) : m_1, ..., m_n \in \mathbb{Z}, m_1 \leq ... \leq m_n \}$. Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$  

Proof. $g \in GL(n, F)$; put $L = gL_0$ where $L_0 = O^n_F$

- $\exists$ basis $v_1, \cdots v_n$ of $L_0$ s.t $\omega^{m_1} v_1, \cdots \omega^{m_n} v_n$ is a basis of $L$.

- $k := [v_1, .., v_n] \in K_0$ and $a = \text{diag}(\omega^{m_1}, ..., \omega^{m_n})$
Let $K_0 = GL(n, O_F)$. Let

$$A^+ = \{ \text{diag}(\omega^{m_1}, ..., \omega^{m_n}) : m_1, ..., m_n \in \mathbb{Z}, m_1 \leq ... \leq m_n \}.$$  

Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$  

Proof. $g \in GL(n, F)$; put $L = gL_0$ where $L_0 = O_F^n$

$\exists$ basis $v_1, \cdots v_n$ of $L_0$ s.t $\omega^{m_1} v_1, \cdots \omega^{m_n} v_n$ is a basis of $L$.

$k := [v_1, .., v_n] \in K_0$ and $a = \text{diag}(\omega^{m_1}, ..., \omega^{m_n})$

simple calculation: $k^{-1}ak.L_0 = L$
Cartan Decomposition

**Theorem**

Let $K_0 = GL(n, O_F)$. Let  

$$A^+ = \{ \text{diag}(\omega^{m_1}, \ldots, \omega^{m_n}) : m_1, \ldots, m_n \in \mathbb{Z}, m_1 \leq \ldots \leq m_n \}.$$  

Then, we have the decomposition  

$$GL(n, F) = K_0 A^+ K_0.$$  

**Proof.** $g \in GL(n, F)$; put $L = gL_0$ where $L_0 = O^n_F$  

\[ \exists \text{ basis } v_1, \ldots, v_n \text{ of } L_0 \text{ s.t } \omega^{m_1} v_1, \ldots, \omega^{m_n} v_n \text{ is a basis of } L. \]  

$k := [v_1, \ldots, v_n] \in K_0$ and $a = \text{diag}(\omega^{m_1}, \ldots, \omega^{m_n})$  

simple calculation: $k^{-1}ak.L_0 = L$  

$g^{-1}k^{-1}ak.L_0 = L_0$
Let $K_0 = GL(n, O_F)$. Let $A^+ = \{diag(\omega^{m_1}, ..., \omega^{m_n}) : m_1, ..., m_n \in \mathbb{Z}, m_1 \leq ... \leq m_n\}$. Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$
Cartan Decomposition

**Theorem**

Let $K_0 = GL(n, O_F)$. Let

$$A^+ = \{ \text{diag}(\omega^{m_1}, ..., \omega^{m_n}) : m_1, ..., m_n \in \mathbb{Z}, m_1 \leq ... \leq m_n \}.$$ Then, we have the decomposition

$$GL(n, F) = K_0 A^+ K_0.$$

**Proof.**

Let $g \in GL(n, F)$; put $L = gL_0$ where $L_0 = O^n_F$.

- $\exists$ basis $v_1, \cdots v_n$ of $L_0$ s.t $\omega^{m_1} v_1, \cdots \omega^{m_n} v_n$ is a basis of $L$.
- $k := [v_1, .., v_n] \in K_0$ and $a = \text{diag}(\omega^{m_1}, ..., \omega^{m_n})$
- Simple calculation: $k^{-1}ak.L_0 = L$
- $g^{-1}k^{-1}ak.L_0 = L_0$
- $g^{-1}k^{-1}ak \in K_0$. $\square$
An Application of Cartan Decomposition

\[ H(G, K_0) = \{ f : G \rightarrow \mathbb{C} : \text{compactly supported} \} \]

\[ H_K_0 := H(G, K_0) \text{ is commutative.} \]

Proof. (Gelfand's Trick)

\[ g \mapsto t_g \rightarrow \iota : H_K_0 \rightarrow H_K_0; \quad \phi \mapsto \iota \phi \]

\[ \iota(\phi_1 * \phi_2) = \iota(\phi_2) * \iota(\phi_1) \]

\[ H_K_0 \text{ has basis: } e_{K_0}g_{K_0} \]

Cartan Decomposition \[ \Rightarrow e_{K_0}g_{K_0} \text{ is invariant under } \iota \iota(\phi_1 * \phi_2) = \phi_1 * \phi_2 = \phi_2 * \phi_1 \]
An Application of Cartan Decomposition

\[ \mathcal{H}(G, K_0) = \left\{ f : G \to \mathbb{C} : \begin{array}{l}
(1) \text{ } f \text{ compactly supported} \\
(2) \text{ } f(k_1gk_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{array} \right\} \]
An Application of Cartan Decomposition

\[ \mathcal{H}(G, K_0) = \left\{ f : G \to \mathbb{C} : \begin{array}{l}
(1) f \text{ compactly supported} \\
(2) f(k_1gk_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{array} \right\} \]

\[ \mathcal{H}_{K_0} := \mathcal{H}(G, K_0) \text{ is commutative.} \]
An Application of Cartan Decomposition

\[ \mathcal{H}(G, K_0) = \left\{ f : G \rightarrow \mathbb{C} : \begin{array}{l}
(1) \text{ } f \text{ compactly supported} \\
(2) \text{ } f(k_1 g k_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{array} \right\} \]

\[ \mathcal{H}_{K_0} := \mathcal{H}(G, K_0) \text{ is commutative.} \]

- Proof. (Gelfand’s Trick)
An Application of Cartan Decomposition

$\mathcal{H}(G, K_0) = \left\{ f : G \rightarrow \mathbb{C} : \begin{align*}
(1) & \quad f \text{ compactly supported} \\
(2) & \quad f(k_1gk_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{align*} \right\}$

$\mathcal{H}_0 := \mathcal{H}(G, K_0)$ is commutative.

- Proof. (Gelfand’s Trick) $g \mapsto t^g \sim \iota : \mathcal{H}_0 \mapsto \mathcal{H}_0$;
An Application of Cartan Decomposition

\[ H(G, K_0) = \begin{cases} f : G \rightarrow \mathbb{C} : & \text{(1) } f \text{ compactly supported} \\ f(k_1gk_2) = f(g) & \forall k_1, k_2 \in K_0, g \in G \end{cases} \]

\[ H_{K_0} := H(G, K_0) \text{ is commutative.} \]

- Proof. (Gelfand’s Trick) \( g \mapsto t^g \sim \iota : H_{K_0} \hookrightarrow H_{K_0} ; \)
- \( \phi^t : \phi^t(g) = \phi( t^g ) ; \)
$\mathcal{H}(G, K_0) =$
\[
\left\{ f : G \to \mathbb{C} : \begin{array}{l}
(1) \text{ } f \text{ compactly supported} \\
(2) \text{ } f(k_1 g k_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{array} \right\}
\]
$\mathcal{H}_{K_0} := \mathcal{H}(G, K_0)$ is commutative.

- Proof. (Gelfand’s Trick) $g \mapsto t^g \leadsto \iota : \mathcal{H}_{K_0} \mapsto \mathcal{H}_{K_0}$;
- $\phi^t : \phi^t(g) = \phi(t^g)$;
- $\iota(\phi_1 * \phi_2) = \iota(\phi_2) * \iota(\phi_1)$
\[ \mathcal{H}(G, K_0) = \left\{ f : G \rightarrow \mathbb{C} : \begin{align*}
& (1) \ f \text{ compactly supported} \\
& (2) \ f(k_1 g k_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{align*} \right\} \]

\[ \mathcal{H}_{K_0} := \mathcal{H}(G, K_0) \text{ is commutative.} \]

- Proof. (Gelfand’s Trick) \( g \mapsto t g \mapsto \iota : \mathcal{H}_{K_0} \mapsto \mathcal{H}_{K_0} ; \)
- \( \phi^t : \phi^t(g) = \phi(t g) ; \)
- \( \iota(\phi_1 * \phi_2) = \iota(\phi_2) * \iota(\phi_1) \)
- \( \mathcal{H}_{K_0} \) has basis: \( e_{K_0 g K_0} \)
\[ \mathcal{H}(G, K_0) = \left\{ f : G \to \mathbb{C} : \begin{array}{l}
(1) \text{ } f \text{ compactly supported} \\
(2) \text{ } f(k_1gk_2) = f(g) \forall k_1, k_2 \in K_0, g \in G
\end{array} \right\} \]

\[ \mathcal{H}_{K_0} := \mathcal{H}(G, K_0) \text{ is commutative.} \]

- **Proof.** (Gelfand’s Trick) \( g \mapsto \iota(g) \mapsto \iota : \mathcal{H}_{K_0} \mapsto \mathcal{H}_{K_0}; \)
- \( \phi^t : \phi^t(g) = \phi(\iota(g)) \)
- \( \iota(\phi_1 \ast \phi_2) = \iota(\phi_2 \ast \iota(\phi_1)) \)
- \( \mathcal{H}_{K_0} \text{ has basis: } e_{K_0gK_0} \)
- **Cartan Decomposition** \( \Rightarrow \) \( e_{K_0gK_0} \text{ is invariant under } \iota \)
An Application of Cartan Decomposition

\[ \mathcal{H}(G, K_0) = \begin{cases} f : G \to \mathbb{C} : \\ (1) f \text{ compactly supported} \\ (2) f(k_1 g k_2) = f(g) \forall k_1, k_2 \in K_0, g \in G \end{cases} \]

\[ \mathcal{H}_{K_0} := \mathcal{H}(G, K_0) \text{ is commutative.} \]

- Proof. (Gelfand’s Trick) \[ g \mapsto t g \sim \iota : \mathcal{H}_{K_0} \hookrightarrow \mathcal{H}_{K_0}; \]
- \[ \phi^t : \phi^t(g) = \phi(t g); \]
- \[ \iota(\phi_1 * \phi_2) = \iota(\phi_2) * \iota(\phi_1) \]
- \[ \mathcal{H}_{K_0} \text{ has basis: } e_{K_0 g K_0} \]
- **Cartan Decomposition** \[ \Rightarrow e_{K_0 g K_0} \text{ is invariant under } \iota \]
- \[ \iota(\phi_1 * \phi_2) = \phi_1 * \phi_2 = \phi_2 * \phi_1 \]
An Application of Cartan Decomposition

\[ \mathcal{H}(G, K_0) = \{ f : G \to \mathbb{C} : \begin{array}{l} (1) \text{ } f \text{ compactly supported} \\ (2) \text{ } f(k_1gk_2) = f(g) \forall k_1, k_2 \in K_0, g \in G \end{array} \} \]

\[ \mathcal{H}_K := \mathcal{H}(G, K_0) \text{ is commutative.} \]

- Proof. (Gelfand's Trick) \( g \mapsto t^g \mapsto \iota : \mathcal{H}_K \mapsto \mathcal{H}_K ; \)
- \( \phi^t : \phi^t(g) = \phi(t^g) ; \)
- \( \iota(\phi_1 * \phi_2) = \iota(\phi_2) * \iota(\phi_1) \)
- \( \mathcal{H}_K \text{ has basis: } e_{K_0gK_0} \)
- **Cartan Decomposition** \( \implies e_{K_0gK_0} \text{ is invariant under } \iota \)
- \( \iota(\phi_1 * \phi_2) = \phi_1 * \phi_2 = \phi_2 * \phi_1 \)
Iwahori Factorization

Let $m \geq 1$, then $K_m = \{ g \in GL(n, \mathbb{O}_F) | g \in I_n + \omega_m M_n(\mathbb{O}_F) \}$.

$K_m$ is an open compact subgroup of $GL(n, F)$ gives a basis of neighborhoods at the identity called principal congruence subgroups.

$U = \begin{pmatrix} 1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & 1 & * \\ \end{pmatrix}$,

$U_- = \begin{pmatrix} 1 & \cdots & * & \cdots & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & 1 & \cdots & * \\ \end{pmatrix}$

Theorem

For $m \geq 1$, we have $K_m = (K_m \cap U_-)(K_m \cap T)(K_m \cap U)$. 

Venketasubramanian. C. G. Ben-Gurion University of the Negev

Structure of p-adic groups
m \geq 1, K_m = \{ g \in GL(n, O_F) | g \in l_n + \omega^m M_n(O_F) \}
- \( m \geq 1, \ K_m = \{ g \in GL(n, O_F) | g \in I_n + \omega^m M_n(O_F) \} \)
- \( K_m \) open compact subgroup of \( GL(n, F) \) gives a basis of neighbourhoods at the identity
• $m \geq 1, K_m = \{ g \in GL(n, O_F) | g \in I_n + \omega^m M_n(O_F) \}$
• $K_m$ open compact subgroup of $GL(n, F)$ gives a basis of neighbourhoods at the identity
• called principal congruence subgroups
\( m \geq 1, \ K_m = \{ g \in GL(n, O_F) | g \in I_n + \omega^m M_n(O_F) \} \)

- \( K_m \) open compact subgroup of \( GL(n, F) \) gives a basis of neighbourhoods at the identity

- called principal congruence subgroups

\[
U = \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}, \quad U^- = \begin{pmatrix} 1 \\ * & \ddots \\ * & * & 1 \end{pmatrix}
\]
Iwahori Factorization

- \( m \geq 1, K_m = \{ g \in GL(n, O_F) | g \in I_n + \omega^m M_n(O_F) \} \)
- \( K_m \) open compact subgroup of \( GL(n, F) \) gives a basis of neighbourhoods at the identity
- called principal congruence subgroups

\[
U = \begin{pmatrix}
1 & * & * \\
 & \ddots & * \\
 & & 1
\end{pmatrix}, \quad U^- = \begin{pmatrix}
1 \\
* & \ddots \\
* & * & 1
\end{pmatrix}
\]

- \( T \) diagonal torus
• $m \geq 1$, $K_m = \{ g \in GL(n, O_F) | g \in l_n + \omega^m M_n(O_F) \}$

• $K_m$ open compact subgroup of $GL(n, F)$ gives a basis of neighbourhoods at the identity

• called principal congruence subgroups

$$U = \begin{pmatrix} 1 & * & * \\ \cdot & \cdot & * \\ \cdot & \cdot & 1 \end{pmatrix}, \quad U^- = \begin{pmatrix} 1 \\ * & \cdot & \cdot \\ * & * & 1 \end{pmatrix}$$

• $T$ diagonal torus

**Theorem**

*For $m \geq 1$, we have*

$$K_m = (K_m \cap U^-)(K_m \cap T)(K_m \cap U)$$
More generally, if $P = MN$ is a parabolic subgroup

- $P^-$ opposite parabolic
More generally, if $P = MN$ is a parabolic subgroup

- $P^-$ opposite parabolic
- $N^-$ unipotent radical of $P^-$
More generally, if $P = MN$ is a parabolic subgroup

- $P^-$ opposite parabolic
- $N^-$ unipotent radical of $P^-$

**Theorem**

$$K_m = (K_m \cap N^-)(K_m \cap M)(K_m \cap N)$$
More generally, if $P = MN$ is a parabolic subgroup
  
  - $P^-$ opposite parabolic
  - $N^-$ unipotent radical of $P^-$

**Theorem**

$$K_m = (K_m \cap N^-)(K_m \cap M)(K_m \cap N)$$
Applications of Iwahori Factorization

If $\pi$ is admissible $\Rightarrow r M, G(\pi)$ is admissible

Proof.

$P = P_{n_1}, \ldots, P_{n_k}$:

$V \mapsto V N$;

$ETST A(V K_m) = V K_0$;

easy $A(V K_m) \subset V K_0$;

Claim: dim($V K_0$) $\leq$ dim($A(V K_m)$);

$\bar{v} \in V N \mapsto v_1 = z^{-1} v$ where $z = diag(\omega_{m_1} I_{n_1}, \ldots, \omega_{m_k} I_{n_k})$;

If $|m_i - m_j| >> 0$ then $K_m$ fixes $v_1$, $K_0$ fixes $A(v_1)$,

$v_2 := \int K_m A(v_1) \, dk$;

$vol(K_m \cap P) = 1$;

$A(v_2) = \bar{v}_i, 1 \leq i \leq n$ be lin. indep.

Find $v_i$ such that $A(v_i) = \bar{v}_i$ and lin. indep.

□
If \( \pi \) is admissible \( \implies \) \( r_{M,G}(\pi) \) is admissible
If $\pi$ is admissible $\implies r_{M, G}(\pi)$ is admissible

Proof. $P = P_{n_1, \ldots, n_k}$
Applications of Iwahori Factorization

If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1,...,n_k}$

- $A : V \mapsto V_N$;

Venkatasubramanian. C. G. Ben-Gurion University of the Negev | Structure of p-adic groups
If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1,\ldots,n_k}$

- $A : V \mapsto V_N$; ETST $A(V^{K_m}) = V^{K_m}$;
If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1, \ldots, n_k}$

- $A : V \mapsto V_N$; ETST $A(V^{K_m}) = V^{K_m^0}$;
- easy $A(V^{K_m}) \subset V^{K_m^0}$
If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1,\ldots,n_k}$

- $A : V \mapsto V_N$; ETST $A(V^{K_m}) = V^{K^0_m}$;
- easy $A(V^{K_m}) \subset V^{K^0_m}$
- Claim: $\dim(V^{K^0_m}) \leq \dim(A(V^{K_m}))$
If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1,...,n_k}$

- $A : V \hookrightarrow V_N$; ETST $A(V^{K_m}) = V^{K^0_m}$;
- easy $A(V^{K_m}) \subset V^{K^0_m}$
- Claim: $\dim(V^{K^0_m}) \leq \dim(A(V^{K_m}))$
- $\bar{v} \in V_N \leadsto v_1 = z^{-1}.v$ where $z = \text{diag}(\omega^{m_1}l_{n_1},...,\omega^{m_k}l_{n_k})$
Applications of Iwahori Factorization

If \( \pi \) is admissible \( \implies \) \( r_{M,G}(\pi) \) is admissible

Proof. \( P = P_{n_1,...,n_k} \)

- \( A : V \mapsto V_N; \) ETST \( A(V^{K_m}) = V^{K_m^0}; \)
- easy \( A(V^{K_m}) \subset V^{K_m^0}; \)
- Claim: \( \dim(V^{K_m^0}) \leq \dim(A(V^{K_m})); \)
- \( \bar{v} \in V_N \mapsto v_1 = z^{-1}.v \) where \( z = \text{diag}(\omega^{m_1}I_{n_1},...,\omega^{m_k}I_{n_k}) \)
- If \( |m_i - m_j| >> 0 \) then \( K_m^- \) fixes \( v_1, K_m^0, K_m^+ \) fixes \( A(v_1) \)
If \( \pi \) is admissible \( \implies r_{M,G}(\pi) \) is admissible

Proof. \( P = P_{n_1, \ldots, n_k} \)

- \( A : V \mapsto V_N; \) ETST \( A(V^{K_m}) = V^{K_m^0}; \)
- easy \( A(V^{K_m}) \subset V^{K_m^0} \)
- Claim: \( \dim(V^{K_m^0}) \leq \dim(A(V^{K_m})) \)
- \( \bar{v} \in V_N \sim \nu_1 = z^{-1}.v \) where \( z = \text{diag}(\omega^{m_1} I_{n_1}, \ldots, \omega^{m_k} I_{n_k}) \)
- If \( |m_i - m_j| > > 0 \) then \( K_m^- \) fixes \( \nu_1, K_m^0, K_m^+ \) fixes \( A(\nu_1) \)
- \( \nu_2 := \int_{K_m} A(\nu_1) d\kappa; \text{vol}(K_m \cap P) = 1; \)
Applications of Iwahori Factorization

If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1,\ldots,n_k}$

- $A : V \mapsto V_N$; ETST $A(V^{K_m}) = V^{K_m^0}$;
- easy $A(V^{K_m}) \subset V^{K_m^0}$
- Claim: $\dim(V^{K_m^0}) \leq \dim(A(V^{K_m}))$
- $\bar{v} \in V_N \leadsto v_1 = z^{-1}.v$ where $z = \text{diag}(\omega^{m_1}I_{n_1},\ldots,\omega^{m_k}I_{n_k})$
- If $|m_i - m_j| >> 0$ then $K^-_m$ fixes $v_1$, $K^0_m$, $K^+_m$ fixes $A(v_1)$
- $v_2 := \int_{K_m} A(v_1)dk$; $\text{vol}(K_m \cap P) = 1$; $A(v_2) = \bar{v}$
If \( \pi \) is admissible \( \implies \) \( r_{M,G}(\pi) \) is admissible

Proof. \( P = P_{n_1,...,n_k} \)

- \( A : V \mapsto V_N; \) ETST \( A(V^{K_m}) = V^{K_m^0} \);
- easy \( A(V^{K_m}) \subseteq V^{K_m^0} \)
- Claim: \( \dim(V^{K_m^0}) \leq \dim(A(V^{K_m})) \)
- \( \tilde{v} \in V_N \leadsto v_1 = z^{-1} \cdot v \) where \( z = \text{diag}(\omega^{m_1}I_{n_1},...,\omega^{m_k}I_{n_k}) \)
- If \( |m_i - m_j| >> 0 \) then \( K_m^- \) fixes \( v_1 \), \( K_m^0 \), \( K_m^+ \) fixes \( A(v_1) \)
- \( v_2 := \int_{K_m} A(v_1) dk; \) vol(\( K_m \cap P \)) = 1; \( A(v_2) = \tilde{v} \)
- \( \tilde{v}_i, 1 \leq i \leq n \) be lin. indep.
If $\pi$ is admissible $\implies r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1, \ldots, n_k}$

- $A : V \mapsto V_N$; ETST $A(V^K_m) = V^{K^0}_m$;
- easy $A(V^K_m) \subset V^{K^0}_m$;
- Claim: $\dim(V^{K^0}_m) \leq \dim(A(V^K_m))$;
- $\bar{v} \in V_N \mapsto v_1 = z^{-1} \cdot v$ where $z = \text{diag}(\omega^{m_1}I_{n_1}, \ldots, \omega^{m_k}I_{n_k})$
- If $|m_i - m_j| >> 0$ then $K^m$ fixes $v_1$, $K^0_m$, $K^+_m$ fixes $A(v_1)$;
- $v_2 := \int_{K_m} A(v_1) \, dk$; $\text{vol}(K_m \cap P) = 1$; $A(v_2) = \bar{v}$
- $\bar{v}_i, 1 \leq i \leq n$ be lin. indep.
- Find $v_i$ such that $A(v_i) = \bar{v}_i$ and lin. indep. □
If $\pi$ is admissible $\Rightarrow r_{M,G}(\pi)$ is admissible

Proof. $P = P_{n_1, \ldots, n_k}$

- $A : V \mapsto V_N$; ETST $A(V^{K_m}) = V^{K_m}$
- easy $A(V^{K_m}) \subset V^{K_m}$
- Claim: $\dim(V^{K_m}) \leq \dim(A(V^{K_m}))$
- $\bar{v} \in V_N \rightsquigarrow \nu_1 = z^{-1}.v$ where $z = \text{diag}(\omega^{m_1}I_{n_1}, \ldots, \omega^{m_k}I_{n_k})$
- If $|m_i - m_j| >> 0$ then $K^-_m$ fixes $\nu_1$, $K^0_m$, $K^+_m$ fixes $A(\nu_1)$
- $\nu_2 := \int_{K_m} A(\nu_1) dk$; $\text{vol}(K_m \cap P) = 1$; $A(\nu_2) = \bar{v}$
- $\bar{v}_i, 1 \leq i \leq n$ be lin. indep.
- Find $v_i$ such that $A(v_i) = \bar{v}_i$ and lin. indep. $\square$
Thanks for your attention and patience!