A remark on different kinds of problems.

The problems without marking are just simple exercises. Be sure you can do them. For problems we use marking $[P]$ and $(*)$ for the more difficult ones and you are advised to write the solution. The sign $(\nabla)$ marks more challenging and more interesting problems which are related to some interesting subjects. It advised to think about these problems.

For an $l$-space $X$ we consider the algebra $C^\infty(X)$ of locally constant functions on $X$. We always consider $C$ with a discrete topology, so $C^\infty(X)$ is just an algebra of continuous functions on $X$. We denote by $S(X) := C^\infty(X)$ the subalgebra of functions with compact support. This last algebra will play an important role in the theory.

Similarly, given a vector space $V$ over $C$ we consider it in discrete topology and denote by $C^\infty(X,V)$ the space of continuous functions on $X$ with values in $V$ and by $S(X,V)$ the subspace of functions with compact support.

1. (i) Show that $S(X,V) = S(X) \otimes V$
(ii) Show that $S(X \times Y) = S(X) \otimes S(Y)$
(iii) Explain why the analogous statements are not true for $C^\infty$ spaces.

For an $l$-space $X$ we define the space of distributions $Dist(X)$ to be the complete dual space $S(X)^*$. (Standard notation $E(\phi) := \int_X \phi E$). The space $Dist(X)$ is naturally a module over the algebra $S(X)$ (in fact even over the algebra $C^\infty(X)$).

2. (i) Let $K$ be a compact $l$-group. Show that there exists unique $K$-invariant distribution $e_K$ on $K$ such $\int_K e_K = 1$. Show that it is also right $K$-invariant. Moreover, if $K$ acts on an $l$ space $X$ transitively then there exists a unique $K$-invariant distribution on $X$.
(ii) Let $G$ be an $l$-group. Show that there exists unique up to scalar $G$-invariant distribution $\mu_G$ on $G$. Moreover, it could be chosen to be positive (i.e it maps positive functions to positive numbers).

Usually we will fix one such positive distribution $\mu_G$. It is called Haar measure on $G$. It is defined uniquely up to positive scalar.

(iii) We denote by $\mathcal{H}(G)$ the space of locally constant distributions with compact support. Show that a choice of Haar measure $\mu_G$ defines an isomorphism of vector spaces $S(G) \rightarrow \mathcal{H}(G)$ by $\phi \mapsto \phi \mu_G$.

$[P]$ 3. Let $Dist(X)_c$ denote the space of distributions on $X$ with compact support.

(i) Let $G$ be an $l$-group. Given a distribution $E \in Dist(G)_c$ with compact support describe its left convolution action on the space $S(G)$, $(E, \phi) \mapsto E * \phi$.
Consider the adjoint right convolution on the space of distributions $(E', E) \mapsto E' * E$.

Show that this defines a convolution operation on the space $Dist(G)_c$ that defines a structure of associative algebra on the space $Dist(G)_c$. Describe the unit
element of this algebra. Finally, describe the map $\delta : G \to Disc(G)_c$ which is a multiplicative embedding.

(ii) Show that $\mathcal{H}(G)$ is a subalgebra (in fact a two sided ideal) of the algebra $Dist(G)_c$.

(iii) Show that for any smooth representation $(\pi, G, V)$ the action of $G$ on $V$ naturally extends to the action of the algebra $Dist(G)_c$.

**Idempotented algebras and non-degenerate modules**

**Definition.** (i) Let $A$ be an associative $C$-algebra of countable dimension. We say that $A$ is an **idempotented algebra** if it satisfies the following condition

(idem) There exists a sequence of idempotent element $e_i \in A$ such that for any element $a \in A$ we have $e_ia = a = ae_i$ for large enough $i$.

This is a slight generalization of the notion of unital algebra (i.e. algebra that has a unit element).

(ii) If $A$ is an idempotented algebra then an $A$-model $M$ is called non-degenerate if $AM = M$. In other words this means that for any element $m \in M$ we have $e_im = m$ for large $i$.

The category of non-degenerate $A$-modules we denote by $M(A)$.

**4.** Let $A$ be an idempotented algebra. Fix an idempotent $e \in A$ and consider the unital algebra $B = eAe$ (we usually denote this algebra by $A_e$).

(i) Show that the module $P = Ae$ is a projective object of the category $M(A)$.

(ii) Show that its endomorphism algebra $\text{End}(P)$ is naturally isomorphic to the algebra $B^\text{op}$ opposite of $B$.

**5.** (Continuation of previous problem) Define functors $R : M(A) \to M(B)$ and $I : M(B) \to M(A)$ by $R(M) = eM = \text{Hom}(P, M)$ and $I(N) = P \otimes_B N$

(i) Show that the functor $I$ is left adjoint to $R$.

(ii) Show that the functor $I$ is right exact and functor $R$ is exact.

(iii) Show that the canonical morphism $N \to RI(N)$ is an isomorphism.

**6.** (Continuation of previous problem)

(i) Assume $L$ is a simple (=irreducible) $A$-module. Show that the module $R(L)$ is either 0 or irreducible.

(ii) Show that this defines a bijection of the set $\text{Irr}(B)$ with the subset $\text{Irr}(A)_e$ consisting of $A$-modules $L$ such that $eL \neq 0$.

(iii) What further assumption on $e$ is required in order to have an equivalence between categories $M(A)$ and $M(B)$?

**7.** Show that the algebra $\mathcal{H}(G)$ is idempotented (consider idempotents $e_K$ corresponding to open compact subgroups of $G$.)

Show that the natural functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$ is an equivalence of categories.
8. (Kaplansky’s trick.)

(i) Let $T : V \to V$ be an endomorphism of a complex vector space $V \neq \{0\}$. By definition the spectrum of $T$ is the set of numbers $\lambda \in \mathbb{C}$ such that the operator $T - \lambda$ is not invertible. Prove the following general result from linear algebra. 

(* ) Assume that $\dim_{\mathbb{C}} V$ is countable. Let $T : V \to V$ be an endomorphism. Then $T$ has a non-empty spectrum.

Hint: Consider $V$ as a $\mathbb{C}[T]$-module. Show that if $\text{Spec}(T) = \emptyset$ then $V$ admits a structure of a $\mathbb{C}(T)$-module.

(ii) Deduce the following corollary.

Let $T : V \to V$ as before. Then $\text{Spec}(T) = \{0\}$ iff the operator $T$ is locally nilpotent, i.e. for every vector $v \in V$ we have that $T^n(v) = 0$ for large $n$.

Hint: Consider the localization of $V$ at $T$ and apply (i).

[P] 9. Prove the following results

Let $A$ be as above (an idempotented algebra over $\mathbb{C}$).

(i) Schur’s lemma: Let $L$ be an irreducible $A$-module. Then $\text{End}_A(L) = \mathbb{C}$.

(ii) The Jacobson radical of $A$ is defined by

$$\text{Jac}(A) = \cap_{(L, \rho) \in \text{Irr}(A) - \{0\}} \ker(\rho : A \to \text{End}(L))$$

Show that the Jacobson radical of $A$ is contained in the set of nilpotent elements.

More specifically, given $a \in A$ that is not nilpotent (i.e. all powers $a^n$ are not zero.) Show that there exists an irreducible $A$-module $L$ such that $aL \neq 0$.

[P] 10. Show that $\text{Jac}(\mathcal{H}(G)) = 0$ (the Jacobson radical is zero).