Local similarity groups and $\ell^2$-homology

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joint work with Roman Sauer
Motivation

Zero-in-the-spectrum conjecture (Gromov 86’)

Let $\mathcal{M}$ be a closed aspherical Riemannian manifold. Then there exists always a $p \in \mathbb{N}$ such that zero is in the spectrum of the Laplacian

$$\Delta_p : \text{dom}(\Delta_p) \subset L^2\Omega^p(\tilde{\mathcal{M}}) \to L^2\Omega^p(\tilde{\mathcal{M}})$$

acting on square integrable $p$-forms on the universal covering.
stronger version: \( p \in \left[ \frac{\dim(M) - 1}{2}, \frac{\dim(M) + 1}{2} \right] \)
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strong Novikov conjecture $\Rightarrow$ zero-in-the-spectrum conjecture
• stronger version: $p \in \left[\frac{\dim(M)-1}{2}, \frac{\dim(M)+1}{2}\right]$ 
• strong Novikov conjecture $\implies$ zero-in-the-spectrum conjecture 
• false if aspherical is dropped (Farber, Weinberger 01’)

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stronger version: \( p \in \left[ \frac{\text{dim}(M) - 1}{2}, \frac{\text{dim}(M) + 1}{2} \right] \)

strong Novikov conjecture \( \implies \) zero-in-the-spectrum conjecture

false if aspherical is dropped (Farber, Weinberger 01’)

true for 2,3-manifolds, locally symmetric spaces, Kähler hyperbolic, \( \sec(M) \leq 0, \text{asdim}(\pi_1 M) < \infty \)
Algebraic version

Let $G = \pi_1 M$, then

\[ 0 \in \text{spec}(\Delta_p) \iff H_p(G, \ell^2 G) \neq 0 \]
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### Question (drop Poincaré duality)

$G$ type $F$, then there exists $p$ with $H_p(G, \ell^2 G) \neq 0$?
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**Answer (Sauer, T. 13')**

No!
Local similarity groups (Hughes 09’)

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- $\gamma$ similarity: $\exists \lambda > 0 : \forall x, y : d(x, y) = \lambda d(\gamma x, \gamma y)$
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- $\gamma$ local similarity: $\forall x \in X : \exists A, B$ balls : $x \in A, \gamma(A) = B, \gamma : A \to B$ similarity
Definition

Let $X$ be a compact ultrametric space. A similarity structure $\text{Sim}$ on $X$ consists of a finite set $\text{Sim}(B_1, B_2)$ of similarities $B_1 \to B_2$ for each pair of balls such that

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1. $\text{Sim}$ forms a groupoid with objects the balls.
2. $A \subset B_1$ subball and $\gamma \in \text{Sim}(B_1, B_2)$ then $\gamma|_A \in \text{Sim}(A, \gamma(A))$. 

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### Definition (local similarity group)

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In 2012, Dan Farley and Bruce Hughes showed that, under suitable conditions on $\text{Sim}$, the groups $\Gamma = \Gamma(\text{Sim})$ are of type $F_\infty$. 
Vanishing of $\ell^2$-homology

Definition

$\text{Sim}$ is called dually contracting if there is a dually contracting ball in $X$. A ball $DC$ is called dually contracting if there are disjoint subballs $B_1, B_2$ of $DC$ and similarities $DC \to B_i$ in $\text{Sim}$. 

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2. If $H < G$ then $H_k(H, M(H)) = 0 \implies H_k(H, M(G)) = 0$.
3. If $H_p(G_i, M(G_i)) = 0$ for $p \leq n_i$ and $i = 1, 2$ then $H_p(G_1 \times G_2, M(G_1 \times G_2)) = 0$ for $p \leq n_1 + n_2 + 1$. 
Theorem (Sauer, T. 13')

If $X$ is a compact ultrametric space with dually contracting similarity structure $\text{Sim}$ and $M$ nice, then for $\Gamma = \Gamma(\text{Sim})$ we have

$$H_k(\Gamma, M(\Gamma)) = 0 \ \forall k$$
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- $H < G$ then $LH \subset LG$ flat ring extension. ($\Rightarrow \square$)
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- $H_0(\Gamma, L\Gamma) = 0$ iff $\Gamma$ non-amenable. ($\Rightarrow$ 1)
- $H < G$ then $LH \subset LG$ flat ring extension. ($\Rightarrow$ 2)
- $LG_1 \otimes_{\mathbb{Z}} LG_2 \subset L(G_1 \times G_2)$ ring extension. ($\Rightarrow$ 3)
Sketch of proof

Fix a dually contracting ball $DC$. Using ping-pong lemma one can show that $\Gamma(Sim|_{DC})$ contains a free non-abelian subgroup and is therefore non-amenable.
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Define simplicial $\Gamma$-complex $Z$ via a poset $(\mathbb{P}, \leq)$:

- objects: $\mathcal{P} = \{P_1, \ldots, P_k\}$ partition of $X$ into non-empty open closed subspaces (finite union of balls)
- $\leq$: $\mathcal{P} \leq Q$ if $Q$ refines $\mathcal{P}$, i.e. for all $Q \in Q$ there is a $P \in \mathcal{P}$ with $Q \subset P$
- action: $g\{P_1, \ldots, P_k\} := \{g(P_1), \ldots, g(P_k)\}$

one can show: $\mathbb{P}$ is directed $\iff Z$ is contractible
Now let $n \in \mathbb{N}$. Define a $\Gamma$-subcomplex $Z_n \subset Z$ via a subpost $(\mathcal{P}_n, \leq)$:

$$\mathcal{P}_n := \{ \mathcal{P} \in \mathcal{P} \mid \text{at least } n \text{ elements } P \in \mathcal{P} \text{ satisfy } \text{Sim}(DC, P) \neq \emptyset \}$$

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Now there is a spectral sequence \( E^k_{pq} \) with

\[
E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, M(\Gamma)) \Rightarrow H_{p+q}(\Gamma, M(\Gamma))
\]

Let \( \sigma = \mathcal{P}_1 < \ldots < \mathcal{P}_p \) a cell, then observe the normal subgroup of the stabilizer group of \( \sigma \)

\[
\Lambda_\sigma := \{ g \in \Gamma \mid g(P) = P \ \forall P \in \mathcal{P}_p \} \triangleleft \Gamma_\sigma
\]
But $\Lambda_\sigma \cong \prod_{P \in \mathcal{P}_p} \Gamma(\text{Sim}|_P)$. By definition of $\mathbb{P}_n$, at least $n$ of the $\Gamma(\text{Sim}|_P)$ are isomorphic to $\Gamma(\text{Sim}|_{DC})$ and consequently

$$H_0(\Gamma(\text{Sim}|_P), M(\Gamma(\text{Sim}|_P))) = 0$$

by (1). From (3) it follows

$$H_q(\Lambda_\sigma, M(\Lambda_\sigma)) = 0 \quad \forall q \in \{0, \ldots, n - 1\}$$

By (2) we have then

$$H_q(\Lambda_\sigma, M(\Gamma)) = 0 \quad \forall q \in \{0, \ldots, n - 1\}$$

The Hochschild-Lyndon-Serre spectral sequence yields $H_q(\Gamma_\sigma, M(\Gamma)) = 0$ in that range and the spectral sequence from above yields $H_i(\Gamma, M(\Gamma)) = 0$ for $i < n$. Since $n$ was arbitrary, the result follows.
Thank you for your attention.