$\ell^2$-Betti numbers for group theorists
A minicourse in 3 parts – 1st lecture

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First impression

\[ \Gamma \mapsto \beta_0^{(2)}(\Gamma), \beta_1^{(2)}(\Gamma), \beta_2^{(2)}(\Gamma) \ldots \in [0, \infty] \]

\[ \Gamma \curvearrowright X \mapsto \beta_0^{(2)}(X), \beta_1^{(2)}(X), \beta_2^{(2)}(X) \ldots \in [0, \infty] \]

- For \( X = E\Gamma \) we have \( \beta_i^{(2)}(X) = \beta_i^{(2)}(\Gamma) \).
\[ \ell^2\text{-Betti numbers} \]

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- For \( X = E\Gamma \) we have \( \beta_i^{(2)}(X) = \beta_i^{(2)}(\Gamma) \).
- Remember: \( \chi(\Gamma) = \sum_{i\geq 0} (-1)^i \beta_i^{(2)}(\Gamma) \) and \( \beta_0^{(2)}(\Gamma) = 0 \) for infinite \( \Gamma \).
- Most important: is \( \beta_i^{(2)}(\Gamma) = 0 \) or not?
- For instance here: If \( \Gamma \) is finitely presented and residually \( p \)-finite with \( \beta_1^{(2)}(\Gamma) > 0 \), then \( \Gamma \) is large (Lackenby).
ℓ²-Betti numbers

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The founders
\( \ell^2 \)-cohomology

- \( \ell^2 \)-Betti numbers are **some dimension** of a certain cohomology – \( \ell^2 \)-cohomology: 
  \[ \beta_i^{(2)}(\Gamma \acts X) = \dim_{\Gamma}(\bar{H}^{(2)}_i(X)) \]

- We postpone the definition of \( \dim_{\Gamma} \) for a while; important for us is:
  \[ \beta_i^{(2)}(\Gamma \acts X) = 0 \iff \bar{H}^{(2)}_i(X) = 0 \]

- The definition of \( \bar{H}^{(2)}_i(X) \) does not involve the \( \Gamma \)-action.
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**Definition**

- Let \( X \) be a CW complex and \( C^0(X) \xrightarrow{d} C^1(X) \xrightarrow{d} \ldots \) its complex of cellular \( \mathbb{C} \)-valued cochains.

- Consider the sub-complex \( C^0_{(2)}(X) \xrightarrow{d} C^1_{(2)}(X) \xrightarrow{d} \ldots \) of **square-summable cochains** and its cohomology:

\[
H^i_{(2)}(X) = \ker(d)/\text{im}(d)
\]

\[
\bar{H}^i_{(2)}(X) = \ker(d)/\text{clos}(\text{im}(d))
\]
\( \ell^2 \)-cohomology by harmonic cocycles

- \( \Delta = dd^* + d^*d : C^i_{(2)}(X) \to C^i_{(2)}(X) \) Laplace operator
- The \( \Delta \)'s as a chain map is \( \simeq \) to the zero map, i.e. there is a chain homotopy \( h : C^i_{(2)}(X) \to C^{i-1}_{(2)}(X) \) with \( hd + dh = \Delta - 0 \).

\[
\begin{align*}
H^i_{(2)}(X) &= 0 \iff \Delta^i \text{ invertible} \\
\bar{H}^i_{(2)}(X) &= 0 \iff \Delta^i \text{ injective}.
\end{align*}
\]
\( \ell^2 \)-cohomology by harmonic cocycles

\( \Delta = dd^* + d^* d : C^i_{(2)}(X) \to C^i_{(2)}(X) \) **Laplace operator**

\( \text{The } \Delta \text{'s as a chain map is } \simeq \text{ to the zero map, i.e. there is a chain homotopy } h : C^i_{(2)}(X) \to C^{i-1}_{(2)}(X) \text{ with } hd + dh = \Delta - 0. \)

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\]

\( \text{One has } \text{ker}(\Delta) = \text{ker}(d) \cap \text{ker}(d^*) \text{; elements in ker}(\Delta) \text{ are called harmonic.} \)

\( d^* : C^1_{(2)}(X) \to C^0_{(2)}(X) \text{ explicitly:} \)

\[
d^* f(v) = \sum_{e(+) = v} f(e) - \sum_{e(-) = v} f(e)
\]
$\ell^2$-cohomology by harmonic cocycles

\[ \cdots \xrightarrow{c} \cdots \xrightarrow{c} \cdots \xrightarrow{c} \cdots \]

\[ \cdots \xrightarrow{1/4} 1/4 \xrightarrow{1/4} \cdots \]

\[ \cdots 1/2 \xrightarrow{1/2} 1/2 \xrightarrow{1/2} \cdots \]

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Geometric and algebraic viewpoint

Geometric

- The definition of $\tilde{H}^i_{(2)}(X)$ does not involve the $\Gamma$-action. The groups $\tilde{H}^i_{(2)}(-)$ are functorial under bi-lipschitz maps.
- Pansu: If $\Gamma \sim_{QI} \Lambda$, then $\tilde{H}^i_{(2)}(E\Gamma) \cong \tilde{H}^i_{(2)}(E\Lambda)$.
- Thus, the vanishing of the $i$th $\ell^2$-Betti number is a QI-invariant.

Algebraic
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Algebraic

- $H^i_{(2)}(X)$ is just cohomology with twisted coefficients for a free cocompact $\Gamma$-CW complex $X$.
- The isomorphism $H^i_{(2)}(X, \ell^2(\Gamma)) \cong H^i_{(2)}(X)$ is induced by

$$\text{hom}_\Gamma(C_i(X), \ell^2(\Gamma)) \xrightarrow{\cong} C^i_{(2)}(X), \ f \mapsto (\sigma \mapsto f(\sigma)(1)).$$

- If $n$ is the number of equivariant $i$-cells in $X$, then

$$\text{hom}_\Gamma(C_i(X), \ell^2(\Gamma)) \cong \ell^2(\Gamma)^n.$$
The von Neumann dimension $\dim_{\Gamma}$

Finite-dimensional vector spaces

Let $W \subset \mathbb{C}^n$ be a subspace and $\text{pr}_W : \mathbb{C}^n \to \mathbb{C}^n$ be the projection onto $W$. Then

$$\dim_{\mathbb{C}}(W) = \text{tr}_{M_n(\mathbb{C})}(\text{pr}_W).$$
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**Finite-dimensional vector spaces**

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**The von Neumann trace**

Let $L(\Gamma)$ be the algebra of $\Gamma$-equivariant bounded operators on $\ell^2(\Gamma)$ (von Neumann algebra of $\Gamma$). For $T \in L(\Gamma)$ define:

$$\text{tr}_\Gamma(T) = \langle Te, e \rangle_{\ell^2(\Gamma)}.$$

It satisfies $\text{tr}_\Gamma(ST) = \text{tr}_\Gamma(TS)$ and extends to $M_n(L(\Gamma))$.

**Hilbert $\Gamma$-modules**

A **Hilbert $\Gamma$-module** is a Hilbert space with an isometric linear $\Gamma$-action such that there is an $\Gamma$-equivariant isometric embedding $H \hookrightarrow \ell^2(\Gamma)^n$.

$$\dim_\Gamma(H) := \text{tr}_{M_n(L(\Gamma))}(\text{pr}_H).$$
ℓ²-Betti numbers

Definition
Let $X$ be a cocompact free $\Gamma$-CW complex. Its $ℓ²$-cohomology $H_i^{(2)}(X)$ is a Hilbert $\Gamma$-module via the embedding:

$$\tilde{H}_i^{(2)}(X) \cong \ker(\Delta^i) \hookrightarrow ℓ²(\Gamma)^n.$$ 

We define the $ℓ²$-Betti numbers as:

$$\beta_i^{(2)}(X) = \dim_Γ(\tilde{H}_i^{(2)}(X))$$

$$\beta_i^{(2)}(\Gamma) = \dim_Γ(\tilde{H}_i^{(2)}(EΓ))$$

Homology versus cohomology
Alternatively, we could define $\beta_i^{(2)}$ by reduced homology. The Laplace operators are the same for homology and cohomology.

Properties
$ℓ²$-Betti numbers satisfy equivariant homotopy invariance, a Küneth formula, and a Euler-Poincare formula....