Finiteness properties of arithmetic groups II

We are interested in $SL_n \mathbb{F}_q[t^\pm 1]$ but today we will concentrate on $n=2$ and consider $SL_2(\mathbb{Z})$ as a warm-up.

$SL_2(\mathbb{Z})$

$SL_2(\mathbb{Z}) \subset H^2 = \{ x + iy \in \mathbb{C} \mid y > 0 \}$ via

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.
\]

1. This action leaves invariant a tessellation of $H^2$ pictured in FIG 1 (below).

2. As in FIG 2, there is a deformation retraction of the fundamental domain onto a boundary edge.

3. Considering the $SL_2(\mathbb{Z})$ action, this induces an $SL_2(\mathbb{Z})$-equivariant deformation retraction of $H^2$ onto a tree. (inblue)
Conclusion, \textbf{SL}_2(\mathbb{Z}) \text{ is a tree.}

Moreover, this action is

- co-compact
- has finite cell stabilizer (via a direct computation)

Thus Brown's criterion applies and \textbf{SL}_2(\mathbb{Z}) \text{ is of type } F_\infty

(i.e. of type } F_m \text{ for all } m)

Fact: \textbf{SL}_2(\mathbb{Z}) \text{ has a torsion free subgroup of finite index that acts freely co-compactly on a locally finite tree.}

\underline{Morse Theory - picture} (see FIG 2 below).

Level sets for the Busemann function \(\beta(x)\) are pictured.

\[
\beta(x) = \lim_{t \to \infty} d(0, t) - d(x, t)
\]

\(\beta(x) \leq d(x, t) \leq d(0, t)\) for some geodesic to \(\infty\).
**Upshot:** We obtain a $SL_2(\mathbb{Z})$ invariant height function

$$h : H^2 \rightarrow \mathbb{R}$$

and can apply Morse theory.

In particular,

- Vertices of tree $\mapsto$ height function (in black for FIG 2) attains its minimum $m$

i.e. $h^{-1}(\infty, m) =$ Set of vertices

- Only critical pts are where horospheres are tangent.

( In FIG 2, 1 light + dark green illustrates down sets for this critical point. 2 dark green illustrates a down set at a non-critical pt. 3 red arrows indicate that later points are not critical.)

**Conclusion:** $H^2$ is $SL_2(\mathbb{Z})$ -equivariantly equivalent to the tree (blue) by means of this height function.
Such points form \( \mathbb{P}^1(\mathbb{Q}) \)

Boundary is \( \mathbb{P}(\mathbb{R}) \)
For: \( SL_2(\mathbb{F}_q[[t]]) \)

\( \mathbb{F}_q(t) \): global function field

\( \mathbb{F}_q((t^{-1})) \): local function field

\( \mathbb{F}_q[[t]] = O_{\deg 3} \) for \( \deg: \mathbb{F}_q[[t]] \to \mathbb{Z}_{\geq 0} \)

\( \mathbb{F}_q[[t]] = \{ f(t) \mid V(f) \geq 0 \} \) where \( V \neq \deg \)

Other valuations \( V \) are \( p \)-adic valuations i.e.

For \( p(t) \in \mathbb{F}_q[[t]] \) irreducible \( \mapsto p \)-adic valuation for \( p(t) \).

\( Y(\mathbb{Q}_5) \) group scheme

This is an arithmetic gp.

\( S \)-arithmetic ring

\( \deg \) induces a metric on \( \mathbb{F}_q(t) \) via

\[ a_n \to 0 \iff \deg(a_n) \to \infty \]

E.g. \( 1, t^{-1}, t^{-2}, t^{-3}, \ldots \to 0 \)

In particular, \( \mathbb{F}_q(t) \) completed w.r.t. this topology is \( \overline{\mathbb{F}_q((t^{-1}))} \)
We wish to reproduce our argument for \( SL_2(\mathbb{Z}) \) so...

**Task:** Find a metric space \( X \) s.t.

\[
P^1(CF_q((t^{-1}))) = SX
\]

Probably a tree will work... but we need to perform an operation:

\[\text{ENDS} \rightarrow \text{TREE}.\]

We define \( \text{TREE} := \prod_{f \in CF_q((t^{-1}))} (\text{formal geodesics}) / \text{glueing} \)

as in the following picture:

- \( a = \sum_{i \in \mathbb{Z}} q_i t^{-i} \)

- \( b = \sum_{i \in \mathbb{Z}} b_i t^{-i} \)

- \( \text{power series elements} \)

i.e. height of glueing is \( -\deg(a-b) \)

e.g. \( a = \sum_{i \geq 0} 1 \) and \( b = \sum 0 t^i = 0 \)

agree on all negative coefficients

and so are glued up to height 0.
Ex: $F_2 = \{0, 1\}$ gives the tree $a = \sum_{i=0}^{\infty} t^{-i}$

$\begin{align*}
S L_2 \left( \mathbb{F}_q \left( \left( t^{-1} \right) \right) \right) & \subseteq X \\
\text{since} \quad \left( \begin{array}{cc}
1 & * \\
0 & 1 \\
\end{array} \right) \subseteq X \quad \text{extends} \quad \left( \begin{array}{cc}
* & * \\
0 & 1 \\
\end{array} \right) \subseteq X, \text{ etc.} \\
\end{align*}$

Fact: $S L_2 \left( \mathbb{F}_q \left( \left( t^{-1} \right) \right) \right) \subseteq X$ s.t.

$S L_2 \left( \mathbb{F}_q \left( \left( t^{-1} \right) \right) \right) \subseteq \text{IP}[\mathbb{F}_q \left( \left( t^{-1} \right) \right)] = \delta X$

Next: generalize the Morse theory picture

Fact (reduction theory): \exists height function $h$

\begin{align*}
\text{twin trees} & \\
X & \xrightarrow{h} \mathbb{R} \\
s.t. & \\
1) & \text{Vertices} \rightarrow \mathbb{Z}_{\geq 0} \\
2) & a) \ u \in \text{V} \Rightarrow h(v) = h(u) \pm 1 \\
& b) h(u) > 0 \Rightarrow \exists! \text{ neighbor } v \text{ s.t.} \\
& \quad h(v) = h(u) + 1 \\
3) & h \text{ is invariant under the } S L_2 \left( \mathbb{F}_q \left[ t \right] \right) - \text{action}
\end{align*}
Rmk: Notice that \( h(u) > 0 \Rightarrow u \) uniquely determines an end.

(2) c) These ends are rational i.e.

elements of \( \mathbb{P}^1(\mathbb{F}_q(t)) \in \text{end} \).

**Height picture:**

Cor: \( SL_2(\mathbb{F}_{q^3}) \) is not finitely generated.

Pf: Set \( X_r = h^{-1}[0,r^3] \) to filter \( X \). Then, this filtration is compact and all hypotheses of Brown's criterion check.

Consider \( X_0 \). \( \prod_0 (X_0 \leq X_r) \) is not trivial for any \( r \) since for all \( r \exists u,v \in X_0 \) such that the path from \( u \) to \( v \) passes through \( X_{r+1} - X_r \). See \( r = 4 \) case above with \( u,v \) circled in red, we \( X_4 - X_3 \) circled in blue.