Lecture 3: Soergel bimodules, Soergel's conjecture and other conjectures.

(W,S) Coxeter system with length function $\ell : W \to \mathbb{N}$

Bruhat order $\leq$ on W: transitive closure of

$$s_1 \cdots \hat{s}_i \cdots s_k < s_1 \cdots\cdots s_k$$

where $s_i \in S$ and $\ell(s_1 \cdots s_k) = k$.

$\mathcal{A} = \mathbb{Z}[v, v^{-1}]$

$\mathcal{H}$ = Iwahori-Hecke algebra of (W,S)

= $\mathcal{A}$-algebra with basis $\{H_w \mid w \in W\}$ such that

$$H_s H_w = \begin{cases} H_{sw} & sw > w \\ H_{sw} + (v^{-1} - v) H_w & sw < w \end{cases} \qquad (s \in S,\ w \in W)$$

Bar operator on $\mathcal{H}$: $\mathbb{Z}$-linear map $h \mapsto \overline{h}$ with $\overline{v^n H_x} = v^{-n} (H_{x^{-1}})^{-1}$

Standard form on $\mathcal{H}$: $\mathbb{Z}[v,v^{-1}]$-sesquilinear inner product with $(H_x, H_y) = \delta_{xy}$

__Thm__ [KL] $\mathcal{H}$ has a unique basis $\{C_w \mid w \in W\}$ which is "Canonical" (see [Web]) in sense that
(1) $\overline{C_w} = C_w$
(2) $C_w \in H_w + \sum_{y < w} v\mathbb{Z}[v] H_y$
(3) $(C_x, C_y) \in \delta_{xy} + v\mathbb{Z}[v]$

$\{C_w\}$ = __KL-basis__. __Positivity conjectures__: (a) $C_w \in \mathbb{N}[v]$-span$\{H_y\}$.

(b) $C_x C_y \in \mathbb{N}[v,v^{-1}]$-span$\{C_z\}$.

Explaining existence and properties of a canonical basis is a generally difficult problem, but by design the most obvious approach is to look for category $\mathcal{C}$ with an isomorphism

$$\mathcal{H} \xrightarrow{\ \mathcal{E}\ } [\mathcal{C}] = \text{split Grothendieck group}$$

which transfers

$$\begin{aligned}
\text{bar operator} &\longrightarrow \text{duality functor} \\
\text{Standard form} &\longrightarrow \text{graded dimension of } \mathrm{Hom}(\cdot,\cdot) \\
\{C_w\} &\longrightarrow \text{representative set of indecomposable objects } \{B_w\}
\end{aligned}$$

In case when W is Weyl group of complex reductive group G with Borel subgroup B, original proof of KL-conjectures described such an isomorphism with $\mathcal{C}$ given by additive and grading closure of semisimple complexes in

$$D^b_{B\times B}(G,\mathbb{C}) = \text{equivariant derived category of } B\text{-biequivariant sheaves on } G.$$

Soergel [S92] found a more elementarily defined category which serves the same purpose and which makes sense for any Coxeter system. This will be the category of $\boxed{\text{Soergel bimodules}}$

All categories of interest will be full subcategories of $\underline{R\text{-Bim}} \overset{\text{def}}{=} \{\mathbb{Z}\text{-graded } R\text{-bimodules } M = \bigoplus_{i \in \mathbb{Z}} M^i$ w/ graded homomorphisms$\}$

where R is some polynomial ring over integral domain $\mathbb{K}$ with characteristic zero.

Given $\mathcal{C} \subset R\text{-Bim}$, define $[\mathcal{C}] =$ abelian group generated by symbols $[M]$ for $M \in \mathcal{C}$ subject to relations $[M] = [A] + [B]$ if $M \cong A \oplus B$

Write $\otimes$ for $\otimes_R$. Then $[\mathcal{C}]$ is ring wrt. $[M][M'] \overset{\text{def}}{=} [M \otimes M']$
(if closed under $\otimes$)

In turn, $[\mathcal{C}]$ becomes $\mathcal{A}$-algebra by setting $v^d[M] \overset{\text{def}}{=} [M(d)]$ where $M(d)$ is grading shift such that $M(d)^i = M^{i+d}$.

<u>Example</u> (Soergel bimodules for $W = S_2$)

Let $W = S_2 = \{1, s\}$
$S = \{s = (1,2)\}$
$R = \mathbb{R}[x]$

W acts on R by $(s \cdot f)(x) = f(-x)$

Let $R^s =$ s-invariants $= \mathbb{R}[x^2]$

(strange convention) $\rightarrow$ Grade R and R^s so that x^n has degree $2n$

Define R-bimodules

$$B_1 \overset{\text{def}}{=} R$$

$$B_s \overset{\text{def}}{=} R \otimes_{R^s} R\,(1)$$

$$B_{[k]} \overset{\text{def}}{=} \underbrace{R \otimes_{R^s} R \otimes_{R^s} \cdots \otimes_{R^s} R}_{k+1 \text{ factors}} \underbrace{(k)}_{\text{grading shift down}}$$

<u>Observe</u> $B_1 = B_{[0]}$ and $B_s = B_{[1]}$ and $B_{[k]} \cong \underbrace{B_s \otimes \cdots \otimes B_s}_{k \text{ factors}}$

$1 \in B_1$ has degree 0

$1 \otimes_{R^s} 1 \in B_s$ has degree -1

$1 \otimes_{R^s} \cdots \otimes_{R^s} 1 \in B_{[k]}$ has degree $-k$

$x \otimes_{R^s} x^2 = x^3 \otimes_{R^s} 1 \in B_s$ has degree 5

<u>Claim</u> (1) B_1 and B_s are indecomposable

(2) $B_{[2]} \cong B_s(1) \oplus B_s(-1)$

<u>Pf.</u> (1) B_1 and B_s are generated as R-bimodules by single elements 1 and $1 \otimes_{R^s} 1$ of lowest degree.

(2) B_s is direct sum of sub-bimodules generated by

$\alpha = 1 \otimes_{R^s} 1 \otimes_{R^s} 1$ in degree -2

$\beta = 1 \otimes_{R^s} x \otimes_{R^s} 1$ in degree 0

Check $\langle \alpha \rangle \cong B_s(1)$ and $\langle \beta \rangle \cong B_s(-1)$. $\square$

Let SBim $\subset$ R-Bim be full subcategory formed by additive, grading closure of bimodules $B_{[K]}$

Facts

(a) SBim closed under $\otimes$ so [SBim] is $\mathcal{A}$-algbra.

(b) Indecomposable objects (up to grading shift and isomorphism) are B_1 and B_s.

(c) Map

$$\mathcal{H} \longrightarrow [SBim]$$
$$C_w \longmapsto [B_w]$$

is isomorphism.

Pf (a) Note that $B_{[K]} \otimes B_{[\ell]} = B_{[K+\ell]}$

(b) Clear from claim.

(c) Recall $C_1 = 1$ and $C_s = H_s + v$.

clearly $C_1 C_w = C_w$ and $B_1 \otimes B_W \cong B_w$.

Check that $C_s C_s = (v+v^{-1}) C_s$ and note that

$$[B_s][B_s] = [B_s \otimes B_s] = [B_s(1) \oplus B_s(-1)] = (v+v^{-1})[B_s].$$

$\square$

SBim for general (W,S): define analogous category of bimodules $B_{[K]}$, then pass to Karoubian envelope.

Setup in detail:

- Let $\mathfrak{h}$ be geometric repn of W defined over commutative ring $\mathbb{K}$, or more generally let $\mathfrak{h}$ be any "realization" of W in sense of [EW2, Def. 3.1].

- Let R be graded ring of polynomial functions

$$R = \bigoplus_{n \geq 0} S^n(\mathfrak{h}^*)$$

with standard S_n-action

(E.g., for $W = S_n$, think of R as (quotient of) $\mathbb{R}[x_1, \dots, x_n]$)

Grade R so that constant fns have degree 0
linear fns have degree 2
quadratic fns have degree 4 etc.

- Let R^s = s-invariants = $\{ f \in R \mid f(s \cdot x) = f(x) \}$ for $s \in S$.

Now, given any sequence $\alpha := (s_1, s_2, \dots, s_k)$ with $s_i \in S$ define

$$B_\alpha \stackrel{\text{def}}{=} \underbrace{R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R}_{k+1 \text{ factors}} \underbrace{(k)}_{\text{grading shift down}} \in R\text{-Bim}$$

Note that $B_\alpha \cong B_{s_1} \otimes B_{s_2} \otimes \dots \otimes B_{s_k}$ where $B_s = R \otimes_{R^s} R(1)$.

View elements of B_α as sums of sequences

porous walls

$$\left(f_0 \underset{s_1}{\Big\}} f_1 \underset{s_2}{\Big\{} f_2 \underset{s_3}{\Big\}} \cdots \underset{s_k}{\Big\{} f_k \right)$$

where $f_i \in R$ and you can slide a scalar across porous wall s_i if the scalar is s_i-invariant.

Def. Category SBim of Soergel bimodules is full subcat of R-Bim whose objects are direct sums & grading shifts of

$M \in R\text{-Bim}$ such that $M \oplus N \cong B_\alpha$
for some $N \in R\text{-Bim}$, sequence α.

$\to$ Ie, SBim = Karoubian envelope of additive, grading closure of $\{$bimodules $B_\alpha\}$

$\to$ Ie, indecomposable objects of SBim are direct summands of B_α's (up to grading shift)

If $M, N \in$ SBim then so are $M \oplus N$, $M(j)$, $M \otimes N$ so [SBim] is $\mathcal{A}$-algebra

Soergel's Categorification Thm I. There is a unique isomorphism

$$\mathcal{H} \xrightarrow{\varepsilon} [\mathrm{SBim}] \quad \text{(of } \mathcal{A}\text{-algebras)}$$

such that $\varepsilon(C_s) = [B_s]$ for $s \in S$.

Moreover $\left(\varepsilon^{-1}([M]), \varepsilon^{-1}([N]) \right) = \sum_{j \in \mathbb{Z}} v^j \, \overbrace{\operatorname{Rank} \operatorname{Hom}_0(M(j), N)}^{\text{deg 0 morphisms}}$

<u>Pf Sketch</u> Uniqueness is immediate, as C_s generate $\mathcal{H}$. Checking that $\mathcal{E}$ is homomorphism relatively not difficult; suffices to check certain isomorphisms in SBim corresponding to braid relations of $\mathcal{H}$. Showing that $\mathcal{E}$ is isomorphism follows from classification of indecomposable objects in SBim, to be described next! $\square$

<u>Remark.</u> It is not at all obvious how to describe $\mathcal{E}(C_w)$ for arbitrary $w \in W$, or even if it is indecomposable, though we hope this is true!

<u>Example</u> (Soergel bimodules for $W = S_3$)

Let $W = S_3 = \{1, s, t, st, ts, sts = sts\}$

$S = \{s = (1,2),\ t = (2,3)\}$

$R = \mathbb{R}[x,y,z]$ graded so that $x^i y^j z^k$ has degree $2(i+j+k)$

W acts on R by
$$s \cdot f(x,y,z) = f(y,x,z)$$
$$t \cdot f(x,y,z) = f(x,z,y)$$

Exercises

(a) The following are indecomposable Soergel bimodules:

$$B_1 = R = \langle 1 \rangle$$

$$B_s = \langle 1 \otimes_{R^s} 1 \rangle$$

$$B_t = \langle 1 \otimes_{R^t} 1 \rangle$$

$$B_{st} \stackrel{\text{def}}{=} B_s \otimes B_t = \langle 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \rangle$$

$$B_{ts} \stackrel{\text{def}}{=} B_t \otimes B_s = \langle 1 \otimes_{R^t} 1 \otimes_{R^s} 1 \rangle$$

(b) If $\delta = y - z$ and $\Delta = \delta \otimes_{R^t} 1 + 1 \otimes_{R^t} \delta$ then

$$B_s \otimes B_t \otimes B_s = \underbrace{\langle 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1 \rangle}_{\text{Call this } B_{sts}} \oplus \underbrace{\langle 1 \otimes_{R^s} \Delta \otimes_{R^s} 1 \rangle}_{\cong B_s}$$

(c) If $\delta' = x - y$ and $\Delta' = \delta' \otimes_{R^s} 1 + 1 \otimes_{R^s} \delta'$ then

$$B_t \otimes B_s \otimes B_t = \underbrace{\langle 1 \otimes_{R^t} 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \rangle}_{\text{Call this } B_{tst}} \oplus \underbrace{\langle 1 \otimes_{R^t} \Delta' \otimes_{R^t} 1 \rangle}_{\cong B_t}$$

(d) $B_{sts} \cong B_{tst}$ are indecomposable.

Facts (1) Indecomposable objects of SBim are (up to isomorphism, grading shift)

$$\{B_1, B_s, B_t, B_{st}, B_{ts}, B_{sts}\}$$

(2) Moreover, $\mathcal{E}(C_w) = B_w$.

Soergel [So7] proves the following generalization of Fact (1):

Soergel's categorification theorem II: For each $w \in W$ there exists up to isomorphism a unique indecomposable bimodule B_w which occurs as a direct summand of B_α for any sequence $\alpha = (s_1, \dots, s_k)$ with $k = \ell(w)$ and $s_1 \cdots s_k = w$, and which appears in no $B_{\alpha'}$ with α' shorter sequence. The set $\{B_w \mid w \in W\}$ represents isomorphism classes of all indecomposable objects in SBim (up to grading shift)

Soergel's conjecture is the generalization of Fact (2).

$\boxed{\underline{\text{Soergel's conjecture}}.\quad \mathcal{E}(C_w) = B_w \ \forall w \in W \text{ when } \mathrm{char}(\mathbb{K}) = 0}$

Recall $\mathbb{K} = R_0$ = field of definition of $\mathfrak{h}$.

Elias and Williamson prove this when $\mathbb{K} = \mathbb{R}$ [EW1].

Immediately implies

$$C_x C_y \in \mathbb{N}[v, v^{-1}]\text{-span}\{C_z\}$$

since by construction $B_x \otimes B_y \cong$ sum of grading shifts of B_z's

Conjecture implies much more by means of explicit formula for inverse $\mathcal{E}^{-1}$.

(__Remark__. Conjecture known to be false for some fields $\mathbb{K}$ with positive characteristic. See [EW2, Remark 3.18])

Soergel's characteristic map. $ch : [SB_im] \to \mathcal{H}$.

To define this we introduce __standard bimodules__:

Given $w \in W$ define $R_w \in R\text{-Bim}$ such that

- As left R-module $R_w = R$
- As right R-module $b \bullet r \stackrel{def}{=} w(r) b$ for $b \in R_w$, $r \in R$.

Since W-action on R is repn, $R_w \otimes R_{w'} \cong R_{ww'}$ $\forall w, w' \in W$

(Standard bi-modules are __not__ necessarily Soergel bimodules.)

Given $f = \sum_{i \in \mathbb{Z}} a_i v^i \in \mathbb{N}[v, v^{-1}]$ define

$$(R_w)^{\oplus f} \overset{\text{def}}{=} \bigoplus_{i \in \mathbb{Z}} (R_w(-i))^{\oplus a_i}$$

grading shift up

<u>Prop.</u> (Soergel)

(a) Every $B \in SBim$ has filtration

$$0 = B^0 \subset B^1 \subset \dots \subset B^m = B$$

where $B^i \in R\text{-}Bim$ and $B^i / B^{i-1} \cong (R_{y_i})^{\oplus h_{y_i}}$ for some $y_i \in W$ and $h_{y_i} \in \mathbb{N}[v, v^{-1}]$.

(b) There is a <u>unique</u> such filtration for which $i < j$ implies $y_i < y_j$ in Bruhat order.

Filtration (b) $\overset{\text{def}}{=}$ <u>standard filtration</u> of $B \in SBim$

Characteristic map: $ch : [SBim] \longrightarrow \mathcal{H}$

$$[B] \longmapsto \sum_{y \in W} h_y \cdot v^{\ell(y)} H_y$$

coeffs in standard filtration of B

<u>Example</u> $W = S_2 = \{1, s\}$

$R = \mathbb{R}[x]$ graded with x^n in degree $2n$.

$R^s = \mathbb{R}[x^2]$

$$B_s \overset{\text{def}}{=} R \underset{R^s}{\otimes} R(1) \in SBim$$

(standard)

The Soergel bimodule B_s has filtration

$$0 = B_0 \subset B_1 = \langle x \underset{R^s}{\otimes} 1 + 1 \underset{R^s}{\otimes} x \rangle \subset B_2 = B_s$$

<u>Check</u>: $B_1 / B_0 = B_1 \cong R_1(-1) = (R_1)^{\oplus v}$

$B_2 / B_1 \cong R_s(1) = (R_s)^{\oplus v^{-1}}$ since B_1 is kernel of map

$$\begin{cases} B_s \longrightarrow R(1) \\ f \underset{R^s}{\otimes} g \longmapsto f(s \cdot g) \end{cases}$$

Thus $h_1 = v$

$h_s = v^{-1}$

$\Rightarrow ch(B_s) = H_s + v = C_s$.

Soergel's Categorification Thm III.

Map $ch : [SBim] \to \mathcal{H}$ is inverse to ε.

Soergel's conjecture is thus equivalent to

Conjecture $ch(B_w) = c_w \ \forall w \in W$.

This implies $c_w \in \mathbb{N}[v]\text{-span}\{H_y\}$ since by definition $ch(B_w) \in \mathbb{N}[v, v^{-1}]\text{-span}\{H_y\}$ and we know $c_w \in \mathbb{Z}[v]\text{-span}\{H_y\}$.

Elias and Williamson [EW1, EW2] have proved Soergel's conjecture when $\mathbb{k} = \mathbb{R}$.

In few words: they show that B_x "looks like cohomology of smooth projective variety" and apply Hodge theoretic ideas of de Cataldo and Migliorini.

More generally, Elias and Williamson have developed effective methods of doing computations in SBim, by giving this monoidal category a presentation as a 2-category, and developing a diagrammatic language to describe its morphisms.

What next? Some related open problems (as evidence that there are many others)

Positivity conjectures.

Let $P_{yw} \in \mathbb{Z}[v^{-2}]$ such that $C_w = \sum_{y \in W} v^{\ell(w)-\ell(y)} P_{yw} \cdot H_y$

$h_{xy}^z \in \mathbb{Z}[v, v^{-1}]$ such that $C_x C_y = \sum_{z \in W} h_{xy}^z C_z$

Conjecture

(1) $P_{yw} \in \mathbb{N}[v^{-2}]$.

(2) $P_{yw} - P_{zw} \in \mathbb{N}[v^{-2}]$ if $y \leq z$

(3) $h_{xy}^z \in \mathbb{N}[v, v^{-1}]$

(4) h_{xy}^z symmetric unimodal, ie of the form

$$a v^{-d} + b v^{-d+2} + c v^{-d+4} + \dots + c v^{d-4} + b v^{d-2} + a v^d$$

where $a \leq b \leq c \leq \dots$

(5) Combinatorial invariance (see [Inc]):

$$P_{y,w} = P_{y',w'}$$

whenever intervals $[y, w]$, $[y', w']$ in Bruhat order are isomorphic posets.

Only (1) & (3) are known for all Coxeter systems.

"Twisted" KL-theory

Let $* \in \mathrm{Aut}(W)$ such that $s^* \in S\ \forall s \in S$ and $** = 1$.

Set $I_* = \{ w \in W \mid w^{-1} = w^* \}$

$\mathcal{H}_* = \mathcal{A}\text{-span}\{ a_w \mid w \in I_* \}$ free $\mathcal{A}$-module

Thm (Lusztig and Vogan) [Lu]

(a) $\mathcal{H}_*$ has (left) $\mathcal{H}$-module structure (analogous to $\mathcal{H}$-algebra structure)

(b) There is unique bar operator $a \mapsto \bar{a}$ on $\mathcal{H}_*$ such that

$$\overline{a_1} = a_1 \quad \text{and} \quad \overline{ha} = \bar{h} \cdot \bar{a} \quad \forall\, h \in \mathcal{H},\ a \in \mathcal{H}_*$$

(c) there is unique basis $\{ A_w \mid w \in I_* \}$ of $\mathcal{H}_*$ such that

$$\overline{A_w} = A_w \in a_w + \sum_{y<w} v\mathbb{Z}[v]\, a_y .$$

Can view KL-basis $\{C_w\}$ as special case of $\{A_w\}$ when $W = W' \times W'$ and $*$ interchanges factors.

While $A_w \notin \mathbb{N}[v]\text{-span}\{a_w\}$ there are analogous positivity conjectures and many mysterious questions attached to this construction. Elias and Williamson have announced work generalizing their methods to this twisted context.

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