Lecture 2: From the Kazhdan-Lusztig basis to (everything but) Soergel bimodules

(W,S) Coxeter system with length function $\ell: W \to \mathbb{N}$

E.g. $W = S_n$ and $S = \{(i,i+1) \mid i=1,\dots,n-1\}$ and $\ell(w) = \#$ inversions of w

Let $\mathcal{A} = \mathbb{Z}[v,v^{-1}]$

$\mathcal{H} = \mathcal{H}(W,S) = \mathcal{A}\text{-span}\{H_w \mid w \in W\}$ free $\mathcal{A}$-module

Thm A There is a unique $\mathcal{A}$-algebra structure on $\mathcal{H}$ such that if $s \in S$ and $w \in W$ then

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w) \\ H_{sw} + (v^{-1}-v)H_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

$\mathcal{H}$ with this structure = Iwahori-Hecke algebra of (W,S)

Observe that $H_x = H_{s_1} \cdots H_{s_k}$ if $x = s_1 \cdots s_k$ is reduced expression (that is, $s_i \in S$ and $\ell(x) = k$)

Also, $H_s^{-1} = H_s + (v - v^{-1}) H_1$ for $s \in S$

Thus, H_x invertible $\forall x \in W$

For $p(v) \in \mathcal{A}$ define $\overline{p} = p(v^{-1})$

For $h = \sum_{w \in W} p_w H_w \in \mathcal{H}$ define $\overline{h} = \sum_{w \in W} \overline{p_w}\, H_{w^{-1}}^{-1}$

Prop. $h \mapsto \overline{h}$ is ring involution of $\mathcal{H}$.

Bruhat order $\leq$ on W: partial order such that $x \leq y$

if $y = s_1 \cdots s_k$ is reduced expression and $x = s_{i_1} s_{i_2} \cdots s_{i_m}$ for

some $1 \leq i_1 < i_2 < \cdots < i_m \leq k$. (Note: $x \leq y \Rightarrow \ell(x) \leq \ell(y)$)

Bruhat order = "subword order"

> Thm B [KL] For each $x \in W$ there is unique element $C_x \in \mathcal{H}$ such that $\overline{C_x} = C_x \in H_x + \sum_{y<x} v\mathbb{Z}[v] H_y$

Pf (After [Si])

Let $C_1 = H_1 = 1$

$$C_s = H_s + v \text{ for } s \in S \Rightarrow C_s H_x = \begin{cases} H_{sx} + vH_x & sx > x \\ H_{sx} + v^{-1} H_x & sx < x \end{cases}$$

Assume $\ell(x) \geq 2$ and C_y is given for $y \in W$ with $\ell(y) < \ell(x)$

Choose $s \in S$ with $sx < x$.

Then $C_{sx} \in H_{sx} + \sum_{y<sx} v\mathbb{Z}[v] H_y$

$$\Rightarrow C_s C_{sx} \in H_x + vH_{sx} + \sum_{y<sx} v\mathbb{Z}[v] C_s H_y$$

This means $C_s C_{sx} = H_x + \sum_{y<x} h_y H_y$ for some $h_y \in \mathbb{Z}[v]$.

Define $\boxed{C_x = C_s C_{sx} - \sum_{y<x} h_y(0) C_y}$

By construction, C_x has desired properties.

Remains to check uniqueness.

Suppose $\{C'_x\}$ another set of elements with desired properties.

Then $\overline{C_x - C'_x} = C_x - C'_x \in \sum_{y<x} v\mathbb{Z}[v] H_y = \sum_{y<x} v\mathbb{Z}[v] C_y$

since $C_y = H_y + v\mathbb{Z}[v]\text{-span}\left\{\begin{array}{c}\text{lower terms}\\ \text{in Bruhat}\\ \text{order}\end{array}\right\}$

Bar invariance of $C_x - C'_x$ and C_y implies that

$C_x - C'_x = \sum_{y<x} h_y C_y$ for some $h_y \in v\mathbb{Z}[v]$ with $\overline{h_y} = h_y$.

But only such polynomials are 0, so $C_x = C'_x$. $\square$

$\{C_x \mid x \in W\}$ is basis for $\mathcal{H}$, the $\boxed{\text{Kazhdan-Lusztig basis}}$

Set $P_{yw} = v^{\ell(y)-\ell(w)} h_{yw}$ where $C_w = \sum_y h_{yw} H_y$

These are the $\boxed{\text{Kazhdan-Lusztig polynomials}}$

Properties which can be shown by induction on length:

Prop. (a) $P_{yw} \in \mathbb{Z}[q]$ where $q = v^{-2}$

(b) P_{yw} has constant term 1 if $y \leq w$ and $P_{ww} = 1$.

(c) $\deg_q(P_{yw}) \leq \frac{\ell(w)-\ell(y)-1}{2}$ if $y < w$

(d) $P_{y^{-1},w^{-1}} = P_{yw}$

(e) If $|S| \leq 2$ then all $P_{yw} \in \{0,1\}$

Thm [Polo] If $f \in \mathbb{N}[q]$ then $P_{yw} = 1 + qf(q)$ for some $y, w \in S_n$ for some $n \gg 0$.

Positivity conjectures:

(1) $C_x \in \mathbb{N}[v]\text{-span}\{H_y \mid y \le x\}$

(2) $C_x C_y \in \mathbb{N}[v, v^{-1}]\text{-span}\{C_z\}$

Ie (1) KL polynomials have positive coeffs

(2) KL structure constants have positive coeffs

KL basis is a "canonical basis"

Let V be a free $\mathcal{A}$-module (definitions from [W])

Preconanical structure on V consists of

(a) "bar involution" $h \mapsto \bar{h}$ which is $\mathbb{Z}$-linear, V-antilinear

(b) $\mathbb{Z}[v, v^{-1}]$-sesquilinear inner product $(\cdot,\cdot) : V \times V \to \mathcal{A}$

with $(x, y) = (\bar{y}, \bar{x})$

(c) "Standard basis" $\{a_c\}$ of V with partially ordered index set $(C, <)$, such that $\overline{a_c} \in a_c + \sum_{c' < c} \mathcal{A} \cdot a_{c'}$

A basis $\{b_c\}$ for V is canonical if

(i) $\overline{b_c} = b_c \in a_c + \sum_{c' < c} \mathcal{A} \cdot a_{c'}$

(ii) $(b_c, b_{c'}) \in \delta_{c,c'} + v\mathbb{Z}[v]$ ("almost orthogonality")

Exercise A precanonical structure admits at most one canonical basis.

Pf. Similar to uniqueness argument in proof of Thm B. $\square$

Recall $\overline{\sum_{w \in W} P_w H_w} = \sum_{w \in W} \overline{P_w}\, H_{w^{-1}}^{-1}$ for $P_w \in \mathbb{Z}[v, v^{-1}]$

Define $\omega\left(\sum_{w \in W} P_w H_w\right) = \sum_{w \in W} \overline{P_w}\, H_{w^{-1}}$ $\rightsquigarrow$ this is anti-involution of $\mathcal{H}$

Standard form on $\mathcal{H}$ is $(h, h') \overset{\text{def}}{=} \left(\begin{array}{c}\text{coeff of } C_1 = H_1 = 1 \\ \text{in product } \omega(h)h'\end{array}\right)$

Thm $\{C_w\}$ is canonical basis for $\mathcal{H}$ with precanonical structure afforded by bar operator $h \mapsto \overline{h}$, standard form, and standard basis $\{H_w\}$.

Pf. Checking that $\mathcal{H}$ has precanonical structure is straightforward.

Given Thm B, suffices to check $(C_x, C_y) \in \delta_{xy} + v\mathbb{Z}[v]$.

This follows from these exercises:

(1) $(C_x, C_y) = $ Coeff of 1 in $C_{x^{-1}} C_y$.

(2) $\omega(C_x) = C_{x^{-1}}$

(3) Coeff of 1 in $H_x H_y$ is $\delta_{x, y^{-1}}$. $\square$

Canonical bases arise from "categorifications"

V	$\longleftrightarrow$	split Grothendieck group of category $\mathcal{C}$
bar operator	$\longleftrightarrow$	duality functor
$(\cdot,\cdot)$	$\longleftrightarrow$	graded dimension of $\mathrm{Hom}(\cdot,\cdot)$
$\{b_c\}$	$\longleftrightarrow$	indecomposable objects, up to grading shift, isomorphism
multiplication by v	$\longleftrightarrow$	grading shift
addition	$\longleftrightarrow$	$\oplus$
multiplication	$\longleftrightarrow$	$\otimes$

How to make sense of these associations?

$\mathcal{C}$ additive category (so, has zero object and contains direct sums)

<u>Def.</u> <u>Split Grothendieck group</u> $[\mathcal{C}]$ of $\mathcal{C}$ is abelian group generated by symbols $[M]$ for $M \in \mathcal{C}$ subject to relations

$$[M] = [A] + [B] \quad \text{if} \quad M \cong A \oplus B.$$

- If $\mathcal{C}$ is monoidal then $[\mathcal{C}]$ is ring: $[A][B] \overset{\text{def}}{=} [A \otimes B]$
- If objects are $(\mathbb{Z})$ graded, then $[\mathcal{C}]$ is $\mathbb{Z}[v,v^{-1}]$-algebra:

$$v[A] \overset{\text{def}}{=} [A(1)] \quad \text{where } A(d) \text{ is grading shift } A(d)^i = A^{i+d}$$

$$\text{where } A = \bigoplus_{i \in \mathbb{Z}} A^i.$$

<u>Example</u> (Soergel bimodules of $(W,S) = (\{1\}, \emptyset)$)

$(R = \mathbb{R})$

$V = \mathcal{A} = \mathbb{Z}[v,v^{-1}] \longleftrightarrow \mathcal{C}$ = $\mathbb{Z}$-graded, finitely generated free R-modules, with graded homomorphisms

Bar operator $p \mapsto \overline{p} \longleftrightarrow$ Duality $M \mapsto \underbrace{\mathrm{Hom}^\bullet(M, \mathbb{1})}_{\substack{j^{th} \text{ graded piece} \\ \text{is } \mathrm{Hom}_R(M(j), \mathbb{1}) \cong M^{-j}}} =: \overline{M}$

where $\mathbb{1} = \begin{cases} R & \text{in degree } 0 \\ 0 & \text{other degrees} \end{cases}$

Form $(p,q) = \overline{p}q \longleftrightarrow$ gr Rank $\mathrm{Hom}(M,N)$

$$= \sum_{n \in \mathbb{Z}} v^n \cdot \mathrm{Rank}\ \mathrm{Hom}_R(\overline{M}(n), N)$$

$\{a_c\} = \{b_c\} = \{1\} \longleftrightarrow$ Indecomposable objects / $\cong$, grading shift are $\mathbb{1}$

<u>Fact</u> $V \to [\mathcal{C}]$ defines $\mathcal{A}$-algebra isomorphism
$1 \mapsto [\mathbb{1}]$ which sends bar operator to duality
form to graded rank (Hom)
$\{b_c\}$ to indecomposable objects.

Soergel bimodules (SBim) will categorify $\mathcal{H}$ and its KL-basis analogously. SBim will be a full subcategory of $\mathbb{Z}$-graded R-bimodules, where R is coordinate ring of geometric repn of (W,S). (If $\mathfrak{h}$ = geometric repn, then $R = S(\mathfrak{h}^*)$.)

That is, there will be an $\mathcal{A}$-algebra isomorphism

(see [S2]) [EW1, EW2]

$$\mathcal{H} \xrightarrow{\ \varepsilon\ } [\mathrm{SBim}]$$

such that $\varepsilon(C_s) = [\text{indecomposable object } B_s]$ for $s \in S$.

Soergel's conjecture (recently proved by Elias and Williamson)

concerns (1) whether a particular set of objects $\{B_w \mid w \in W\}$ in SBim, which represent isomorphism classes of all indecomposable objects up to grading shifts, are in fact such that $\varepsilon(C_w) = B_w$.

(2) An amazing formula for the inverse ε^{-1}

(1) will imply positivity of KL structure constants: $C_x C_y \in \mathbb{N}[v,v^{-1}]\text{-span}\{C_z\}$

(2) will imply positivity of KL polynomials: $C_x \in \mathbb{N}[v]\text{-span}\{H_y\}$

References

[KL] Kazhdan and Lusztig, Representations of Coxeter groups and Hecke algebras

[S1] Soergel, Kazhdan-Lusztig polynomials and a combinatoric for tilting modules

[S2] Soergel, Kazhdan-Lusztig polynomials and indecomposable bimodules over polynomial rings.

[W] Webster, Canonical bases and higher representation theory

[EW1] Elias and Williamson, The Hodge theory of Soergel bimodules

[EW2] Elias and Williamson, Generators and relations for Soergel bimodules

[Polo] Polo, Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups.

[EK] Elias and Khovanov, Diagrammatics for Soergel Categories.