Lecture 1: An organic introduction to the Iwahori-Hecke algebra of a Coxeter system and its Kazhdan-Lusztig basis

G finite group

$B \subset G$ subgroup

Let $e = \frac{1}{|B|} \sum_{h \in B} h \in \mathbb{C}B \subset \mathbb{C}G$

Fact $e^2 = e$

Pf. $he = e \quad \forall h \in B$ □

Define $\mathrm{Ind}_B^G(\mathbb{1}) \overset{\text{def}}{=} \mathbb{C}Ge = \mathbb{C}\text{-span}\{ge \mid g \in G\}$

$\mathcal{H} \overset{\text{def}}{=} e\mathbb{C}Ge = \mathbb{C}\text{-span}\{ege \mid g \in G\}$

Remarks $\mathrm{Ind}_B^G(\mathbb{1})$ is the G-module induced from the trivial repn of B.

$\mathcal{H}$ is a $\mathbb{C}$-algebra with unit $e = e1e$.

Prop. [CR §11D] $\mathcal{H} \cong \mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G(\mathbb{1}))$ as $\mathbb{C}$-algebras

$$\underbrace{\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G(\mathbb{1}))}_{\overset{\text{def}}{=} \left\{ \begin{array}{l} \mathbb{C}\text{-linear } \varphi : \mathrm{Ind}_B^G(\mathbb{1}) \to \mathrm{Ind}_B^G(\mathbb{1}) \\ \text{with } \varphi(gx) = g\varphi(x) \ \forall g \in G \end{array} \right\}}$$

Pf. Check that map
$$\mathrm{End}_{\mathbb{C}G}(\mathrm{Ind}_B^G(\mathbb{1})) \to \mathcal{H}, \qquad \varphi \longmapsto \varphi(e)$$
is isomorphism □

Call $\mathcal{H}$ a <u>Hecke algebra</u>. Can construct such a thing whenever we have an algebra (eg $\mathbb{C}G$) and an idempotent e.

Why study $\mathcal{H}$?

Thm [CR §11D]

(1) $\mathcal{H}$ is semisimple and so decomposes as sum of two-sided ideals
$$\mathcal{H} = \bigoplus_{u \in U} \mathcal{H} u \mathcal{H} \text{ where } U \text{ is set of primitive idempotents.}$$

(2) The maps
$$\begin{array}{ccccc} U & \longrightarrow & \{\text{Irr } \mathcal{H}\text{-modules}\}/\cong & \longrightarrow & \left\{\begin{matrix}\text{Irr. } G\text{-submodules} \\ \text{of } \mathrm{Ind}_B^G(\mathbb{1})\end{matrix}\right\}/\cong \\ u & \longmapsto & \mathcal{H}u & \longmapsto & \mathbb{C}Gu \end{array}$$

are bijections, and mult. of $\mathbb{C}Gu$ in $\mathrm{Ind}_B^G(\mathbb{1})$ is $\dim(\mathcal{H}u)$.

Moral: $\mathcal{H}$ is a "nice" algebra, typically smaller and easier to study than $\mathrm{Ind}_B^G(\mathbb{1})$, whose repns give information about the irr. decomp. of induced repns.

How to study $\mathcal{H}$?

Let $W = B \backslash G / B$ denote set of B-double coset reps in G so that $G = \bigsqcup_{w \in W} BwB$

<u>Prop</u> [CR §11D] Let $T_w = \frac{1}{|B|} \sum_{x \in BwB} x$.

Then $\{T_w \mid w \in W\}$ is basis for $\mathcal{H}$, not depending on choice of W.

Call $\{T_w\}$ the <u>standard basis</u> of $\mathcal{H}$.

Motivating example

Let $G = GL_n(\mathbb{F}_q)$

B = subgroup of upper-Δ matrices

S_n = subgroup of permutation matrices $\left\{\begin{pmatrix} 1&0&0\\0&0&1\\0&1&0\end{pmatrix}\right\} \subset G$.

Prop. $S_n = B\backslash G/B$, i.e., S_n is set of B-double coset reps.

Pf. Gaussian elimination. $\square$

Standard basis of $\mathcal{H}$ consists of elements indexed by $w \in S_n$.

Let $s_i = (i, i+1) \in S_n$.

Thm [B, Thm 11] $\quad T_{s_i} T_w = \begin{cases} T_{s_i w} & \text{if } w(i) < w(i+1) \\ q T_{s_i w} + (q-1) T_w & \text{if } w(i) > w(i+1) \end{cases}$

Exercise Check this when $n=2$.

Remarks
- In this case $\mathcal{H}$ is described entirely by S_n and q.
- Such is the case whenever G is finite group of Lie type and $B \subset G$ is its Borel subgroup, but with role of S_n replaced by another Weyl group W; see [Car]

Iwahori: To any Coxeter system (W, S) can attach algebra $\mathcal{H}(W,S)$ of which the Hecke algebras above (with W a Weyl group) are special cases.

Def [BB, Thm 1.5.1] A <u>Coxeter system</u> (W,S) is a group W with a generating set S, such that every $s \in S$ has order two ($s^2 = 1$) and the following "exchange property" holds:

$$\begin{cases} \text{Define } \ell(w) = \text{smallest } k \text{ s.t. } w = s_1 \cdots s_k \text{ for } s_i \in S. \\ \text{If } s \in S,\ w \in W \text{ and } \ell(sw) \le \ell(w), \text{ then} \\ \qquad sw = s_1 \cdots \hat{s}_i \cdots s_k \ \text{(omit one factor)} \end{cases}$$

Elems of W are (equiv classes of) words in alphabet S, without adjacent letters equal.

<u>Examples</u>
$$\begin{cases} W = S_n \\ S = \{(i,i+1) \mid i = 1, \dots, n-1\} \\ \ell(w) = \# \left\{ (i,j) \;\middle|\; \begin{smallmatrix} 1 \le i < j \le n \\ w(i) > w(j) \end{smallmatrix} \right\} \end{cases}$$

$$\begin{cases} W = \text{dihedral group of order } 2n = \left\langle r = \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \\ S = \{s, rs\} \end{cases}$$

$$\begin{cases} W = \text{universal Coxeter group } \langle s_1, \dots, s_n \mid s_i^2 = 1 \rangle \\ S = \{s_i\} \\ \ell(s_1 \cdots s_k) = k \text{ provided } s_i \neq s_{i+1}\ \forall i \end{cases}$$

Let $\boxed{\mathcal{A} = \mathbb{Z}[v, v^{-1}]}$ ring of Laurent polyns in indeterminate v.

Set $\boxed{q = v^{-2}}$

Fix Coxeter system (W, S)

formal symbols

Let $\mathcal{H} = \mathcal{H}(W,S) = \mathcal{A}\text{-span}\{T_w \mid w \in W\}$ (free $\mathcal{A}$-module)

<u>Thm A.</u> There is a unique $\mathcal{A}$-algebra structure on $\mathcal{H}$ such that if $s \in S$ and $w \in W$

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) > \ell(w) \\ q T_{sw} + (q-1) T_w & \ell(sw) < \ell(w) \end{cases}$$

<u>Remarks</u> - Call $\mathcal{H}$ w/this structure the <u>generic Iwahori-Hecke algebra</u>

- Thm holds if we replace q by multiple indeterminates q_s $(s \in S)$ subject to some conditions $\rightsquigarrow$ <u>multiple parameter Iwahori-Hecke algebra</u>

- Thm holds if $\mathcal{A}$ is any commutative ring and $q \in \mathcal{A}$. Resulting algebra is <u>specialization</u> of $\mathcal{H}$, isomorphic to $R \otimes_\phi \mathcal{H}$ where ring homomorphism $\phi : \mathcal{A} \to R$

- If $R = \mathbb{C}$ then specialization is semisimple if $\phi(q)$ is not root of unity, in which case irreps of $\mathbb{C} \otimes_\phi \mathcal{H}$ in bijection with irreps of W.

Change of notation: set $\boxed{H_x = v^{\ell(x)} T_x}$

<u>Facts</u> (a) $H_x = H_{s_1} \cdots H_{s_k}$ if $x = s_1 \cdots s_k$ and $s_i \in S$ and $\ell(x) = k$.

(b) $\{H_x \mid x \in W\}$ basis for $\mathcal{H}$.

(c) $H_s^{-1} = H_s + (v - v^{-1})$ so H_x (always) invertible

<u>Prop.</u> There is a unique ring homomorphism $\mathcal{H} \to \mathcal{H}$, $H \mapsto \overline{H}$

such that $\begin{cases} \overline{v} = v^{-1} \\ \overline{H_x} = (H_{x^{-1}})^{-1} \end{cases}$

<u>Pf.</u> Suffices to check $\overline{H_s H_x} = \overline{H_s}\,\overline{H_x}$ for $s \in S$ and $x \in W$.
This is simple calculation using that

$$(s \in S, x \in W) \quad H_s H_x = \begin{cases} H_{sw} & \ell(sw) > \ell(w) \\ H_{sw} + (v - v^{-1}) H_w & \ell(sw) < \ell(w) \end{cases} \quad \square$$

<u>Thm B</u> [KL] For each $x \in W$ there is a unique $C_x \in \mathcal{H}$ such that $\overline{C_x} = C_x \in H_x + \sum_{y \in W} v\mathbb{Z}[v] H_y$

$\{C_x \mid x \in W\}$ is a basis for $\mathcal{H}$ (not obvious)

Call this the <u>Kazhdan-Lusztig (KL) basis</u>

Define $h_{yx} \in \mathcal{A}$ by $C_x = \sum_y h_{yx} H_y$ for $x, y \in W$

Set $\boxed{P_{yx} = v^{\ell(y) - \ell(x)} h_{yx}}$

Then $P_{yx} \in \mathbb{Z}[q]$ are KL polynomials

Why this basis?

- Connections to repn theory: original KL conjectures proved by Beilinson, Bernstein, Brylinski, Kashiwara (1981) assert that values $h_{yx}(1)$ encode Jordan–Hölder multiplicities of irreducible quotients in Verma modules attached to f.d. ~~ses~~ semisimple Lie algebras / $\mathbb{C}$.

- Mysterious properties:

  Positivity conjectures: $\begin{cases} P_{yx} \in \mathbb{N}[q] \\ C_x C_y \in \sum_z \mathbb{N}[v, v^{-1}] C_z \end{cases}$

  First shown for Weyl groups by interpreting coeffs of P_{yx} as dimensions of intersection homology groups for Schubert varieties of reductive group G.
  However, hold more generally, by recent work of Elias and Williamson, to be described next time!

References

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[Car] Carter, Finite Groups of Lie Type

[CR] Curtis and Reiner, Methods of Representation Theory.

[KL] Kazhdan and Lusztig, Representations of Coxeter Groups and Hecke algebras.

[S] Soergel, Kazhdan-Lusztig polynomials and a combinatoric for tilting modules.