What are \( p \)-adic groups? Where do they show up?

\[ \text{GL}_n(\mathbb{Q}) \] reps related to \( \text{GL}(\mathbb{Q}/\mathbb{Q}) \) reps

Some kinds of reps:
- Algebraic: on \( \mathbb{Q}^n \) by matrix multiplication
- Continuous: restrictions of \( \text{GL}(\mathbb{R}) \) reps, e.g., on \( L^2(\mathbb{R}^n) \) via Fourier
- Numerical: \( \mathbb{Q}^n \times A \to \mathbb{Q} \), \( A \in \mathbb{R} > 0 \)
  \[ \text{where } 1 + p \text{ is the } p \text{-adic absolute value } 1 + p = \frac{1}{p^n} \]

Claim: \( \mathbb{Q}^n \) does not extend continuously to \( \text{GL}(\mathbb{R}) \): orient \( m \to p^m \) and \( c = \lim_{m \to \infty} p^m \).

To find all (irred) reps of \( \text{GL}_n(\mathbb{Q}) \), one has to consider all embeddings

\[ \text{GL}_n(\mathbb{Q}) \supset \text{GL}_n(\mathbb{Q}_p) \supset \text{GL}_n(\mathbb{Q}) \ldots \supset \text{GL}_n(\mathbb{R}) \]

Similarities/Differences between Lie-/p-adic groups: Lie groups are smooth.

What's \( \text{GL}_n(\mathbb{Q}_p) \) as top. space? On \( \text{GL}_n(\mathbb{Q}_p) \): metric of \( (A,B) = \max \{ |a_{ij} - b_{ij}|_p \} \)

Lemmas: \( (\text{GL}_n(\mathbb{Q}_p), \mathcal{O}_p) \) is totally disconnected (connected components are points),

Proof: Let \( Y \subset \text{GL}_n(\mathbb{Q}_p) \) connected, \( \Phi, A \in Y \), \( \mathcal{O}_p, Y \to \mathbb{R}, \quad \text{continous}, \quad B \mapsto \det(A^{-1}B) \)

Proper properties of \( \text{GL}_n(\mathbb{Q}_p) \):
1) Complete metric space (with correct metric)
2) Totally disconnected, not discrete.
3) Locally compact
4) Algebraic (multiplicative and algebraic parts of the entries)
5) Top group with 1, \( p \mathcal{O}_p \).

Examples: \( \text{SO}(\mathbb{Q}_p) \), \( \text{SL}(\mathbb{Q}_p) \), \( \text{Spin}(\mathbb{Q}_p) \)

Comparison w/ Lie groups:

- \( \text{SU}(2), \text{SU}(3) \), sometimes \( 4, 5 \), never \( 2 \) connected simply.

Lie groups arise often as symmetries of something: \( \text{On}(\mathbb{R}) = \{ \text{isometries of } \mathbb{R}^n \} \)

\( \text{PGL}_2(\mathbb{R}) = \{ \text{isometries of } \mathbb{H} \} \)

\( p \)-adic groups arise as isometries of \( \mathbb{F}_p \) trees

\( p = 2 \), \# lines in \( \mathbb{F}_p^2 \),

\[ \mathbb{F}_p^2 \]

\( p = 3 \)

\( p = 2 \), \# lines in \( \mathbb{F}_p^2 \),

\[ \mathbb{F}_p^2 \]

\( R = \mathbb{F}_p [1 + 1] \), \# fields \# \( R - \text{submodules of } \mathbb{F}_p^2 \) of index \( R \).

\( \text{Submodules of } \mathbb{L} \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R} = \mathbb{F}_p^2 \)
neighborhoods of \( L \): R-submodules of \( L \) of index \( p \) modulo \( L \sim L \forall n \in \mathbb{N} \\
\Rightarrow \) there exists a bijection \( \{ \text{vertices of tree} \} \leftrightarrow \{ \text{R-submodules of} \ L \text{ of index} \ p^n \} \)

\( \Rightarrow \) \( GL_2(R) \) acts on tree

Replace \( R \) by quotient field \( F = \mathbb{F}_p \left[ \left\{ t^n \right\} \right] = \mathbb{F}_p (\langle t \rangle) \)
Replace submodules of \( R^2 \) by \( R \)-lattices in \( F^2 \)
Lattice in \( F^n \): compact + discrete quotients, \( R \)-submodule (same as \( RV \cup RW \), \( V, W \in F^2 \) lin. indep.)

Homology of lattices \( L \in L^1 \Rightarrow L = cL^1 \) for some \( a \in \mathbb{F}_p^* \\
\Rightarrow \) bijection \( \{ \text{vertices of tree} \} \leftrightarrow \{ \text{R-lattices in} \ F^2 \}/R \\
\Rightarrow \) \( GL_2(F) \) acts on \( T \)

Stabilizer of \( L^2 \) is \( GL_2(R) \cdot Z(\mathbb{G}_1(F)) \)
Valuated fields: \[ \text{Def: } \text{An absolute value on a field } F \text{ is a map } v: F \to \mathbb{R}_{\geq 0} \text{ s.t. } v(xy) = v(x)v(y) \text{ and } v(x+y) \geq \min\{v(x), v(y)\} \forall x, y \in F. \]

Examples: \( v(1,1) = 1, v(x) = |x|, v_1(x) = |x|, v_2(x) = \max\{|x|, |y|\}, v_3(x) = \|x\|_1, v_4(x) = \max\{|x|, |y|\}. \)

\( v(x,y) = 1 \text{ for } x \neq 1 \text{ in } \mathbb{R}. \)

- metric on \( F \), \( d(x,y) = |x-y| \)

\[ \text{Def: A discrete valuation on } F \text{ is a surjective map } F^* \to \mathbb{Z} \text{ s.t. } v(x+y) = v(x) + v(y) \text{ and } v(xy) = v(x)v(y) \forall x, y \in F^*. \]

Examples: \( F = \mathbb{Q}_p, v(p^n) = n \) if \( x \in \mathbb{Z}_p \setminus p\mathbb{Z}_p. \)

\( F = \mathbb{Q}(x), E \text{ field, } v(x^n) = v(x) = n \text{ if } an \text{ is } \alpha. \)

\( \mathcal{O} = \{ x \in F : v(x) > 0 \} \) subring of \( F. \)

\( \mathcal{O}^* = \{ x \in F : v(x) = 0 \} \) invertibles in \( \mathcal{O}. \)

\( m = \{ x \in F : v(x) \geq 0 \} \) unique maximal ideal in \( \mathcal{O}. \)

\( \mathcal{O}_m \text{ residue field of } F. \)

For \( E \mathcal{O}: \) \( \mathcal{O} = \mathbb{Z}_p, m = p\mathbb{Z}_p, \mathcal{O}_m = \mathbb{F}_p, \)

For \( E_{\mathcal{O}}(E) = E_{\mathcal{O}}[E_{\mathcal{O}}], m = E_{\mathcal{O}}[E_{\mathcal{O}}], \mathcal{O}_m = E. \)

\( 1 \mapsto e^{-v(0)} \) is a non-archimedean absolute value on \( F(\text{con class field number} > 1) \) instead of \( e \)

\( \rightarrow \) topology on \( F. \)

An element \( \pi \in F \) \( v(\pi) = 1 \) is called a \( \text{uniformizer} \) of \( F. \)

\( \pi^0 \mathcal{O} = \{ x \in F : v(x) > 0 \} \) ball around \( \mathcal{O} \) of radius \( e^{-v(0)} \)

\( = B(0, e^{-v(0)}). \)

\( \{ \pi^0 \mathcal{O} : \text{new} \} \) basis of neighborhoods of \( \mathcal{O} \).

Prop: \( \pi^0 \mathcal{O} \) is compact \( \Rightarrow \mathcal{O} \) complete wrt. \( \pi^0 \mathcal{O} \) and \( \mathcal{O}_m \) finite.

Proof: A metric space \( X \) is compact \( \Rightarrow X \) is complete and totally bounded. \( v(x) = 0 \) \( \Rightarrow x = \sum x_i. \)

\( e^{-v(0)} \) Suppose \( \pi^0 \mathcal{O} \) compact \( \Rightarrow \mathcal{O} \) compact (\( \pi^0 \mathcal{O} \) is homeomorphism).

\( \Rightarrow \) Any \( \mathcal{O} \) will finitely many balls of radius \( \pi^0 \mathcal{O}, B(x, \pi^0 \mathcal{O}) = x + \pi^0 \mathcal{O}. \)

\( \Rightarrow \mathcal{O}_m \text{ is } \mathcal{O}_m \text{ finite.} \)

Suppose \( \mathcal{O} \) complete, \( \mathcal{O}_m \) finite.

Let \( \mathcal{O} \), \( \mathcal{O}_m \text{ complete } \)

\( \Rightarrow x_i \rightarrow x_{\mathcal{O}_m} \pi^0 \mathcal{O} = B(x, \pi^0 \mathcal{O}) \mathcal{O}_m \).
In particular, \( \mathbb{Q} = \mathbb{Z}_p \) is compact (because \( \mathbb{Q} \) is the completion of \( \mathbb{Q} \) wrt. \( 1/p \)),

\( \mathbb{Q} = \mathbb{E}[F] \) compact \( \Rightarrow \) \( E \) finite.

(We had a similar trick last time I talked to you, just can't use local compactness anymore.)

**Local fields:**

**Def:** A local field is a topological field which is locally compact but not discrete.

**Examples:** \( \mathbb{Q}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_q(e) \)

**Def:** (normalized absolute value) Let \( \mu \) be a measure on \( F \) and \( x \) be a positive element of \( F \).

\[ A \mapsto \mu(x A), \quad x \in F^\times, \quad \text{is again a Haar measure} \Rightarrow \exists x \in F^\times: (\forall y \in F^\times) \mu(x A) = \mu(y A) \]

\( 1 \cdot |.| \) is an absolute value on \( F \).

**Example:** \( F = \mathbb{Q}_p \), \( A = \mathbb{Z}_p \), \( x \in A \).\)

\[ |A| = |x A| = \mu(x A) = \mu(1) \cdot |x| \]

\[ |A| = |x A| = \mu(x A) = \mu(1) \cdot |x| \]

\[ \Rightarrow 1 \cdot |x|_p = 1 \cdot |x|_p. \]

**Lemma:** Let \( E \) be a local field, \( E/F \) finite extension \( \Rightarrow E \) also a local field.

**Proof:** \( E \cong F^n \) as \( F \)-vector space, so \( E \) and the product topology from \( F^n \) is locally compact.

**Exercise:** Every \( x \in E \) induces a \( F \)-linear map \( E \to E \) with determinant \( \det_E(x) \).

**Examples:** \( 1 \cdot |.| = \det(E/F) \).

**Special cases:** \( E = \mathbb{C} : |z|_{E/C} = 1 \), \( 1 \cdot |.| = 1 \cdot |.| \).

**Theorem:** (Classification of local fields)

Let \( F \) be a local field. Then \( F \) is archimedean or non-archimedean.

**a)** If \( F \) is archimedean \( \Rightarrow \) \( F = \mathbb{R}, \mathbb{C} \).

**b)** If \( F \) is non-archimedean \( \Rightarrow \exists \) discrete valuation.

If \( \mathrm{char}(F) = 0 \) \( \Rightarrow \) \( F \) is finite extension of \( \mathbb{Q} \) for some \( \mathbb{Q} \).

If \( \mathrm{char}(F) \neq 0 \) \( \Rightarrow \) \( F \cong \mathbb{F}_q(e) \) with \( \mathbb{F}_q \) finite field.
Def: A *pradic* field is a nonarchimedean local field with residual characteristic $p$.

The multiplicative group of a pradic field

$v: F^* \to \mathbb{Z}$ is continuous, so $O^* = \{ x \in F^* : v(x) = 0 \}$ open and closed in $F^*$

$F^* \cong O^* \times \mathbb{Z}$

$u^m = - (\alpha, m)$, $\pi$ uniformizer

Lemma: $O^*$ is the unique maximal compact subgroup of $F^*$.

Proof: Suppose $K \subset F^*$ compact $\Rightarrow \nu(K) \leq \mathbb{Z}$ compact subgroup $\Rightarrow \nu(K) = \{0\}$

$O$ is compact (above Prop. + classification or use that $\mathfrak{m}O : \{ \text{maximal} \}$ is a neighborhood basis of $O \subset F$, $F/\{\text{maximal} \} \cong \mathfrak{m}O$ compact $\Rightarrow \mathfrak{m}O$ compact $\Rightarrow O$ compact)

$m$ is compact, $O/m$ also open $\Rightarrow O/m$ finite $\Rightarrow O^* = O/m$ open and closed

Since $O$ is compact Hausdorff, so is $O^* \cong O$.

Linear algebraic groups: Examples: $G_{m}, G_{a}, G_{l}$, $\text{Spec}(F) = \{ \mathbf{A} \in M_{n}(F) : A^t \alpha A = \alpha \}$

$G_{m}(F) = F^*, G_{a}(F) = F$ multiplicative/additive groups.

Def: An algebraic group $G$ is an algebraic variety (over $F$)

- a group
- multiplication and inverse are morphisms of algebraic varieties.

$G$ is linear if it is a subgroup of $G_{n}$ for some $n$.

$J G := \{ f \in F[G, \text{det}^{-1}] \mid f \bigg|_G = 0 \}$

If $G$ is generated by $F[G, \text{det}^{-1}] \cap J G \Rightarrow G$ defined over $F$.

In this case, the group of $F$-rational points of $G$ is $G(F) = \{ \text{all } f \in G(F) : f(1) = 0 \}$

$\forall f \in F[G, \text{det}^{-1}] \cap J G$
Example: Unitary groups: $E/F$ field extension of degree $2$, char $(F) = 2$

$\Rightarrow$ Galois extension with Galois group $\text{Gal}(E/F) = \{\text{id}_E, \sigma\}$

$\Rightarrow$ $\text{U}_n(E/F) = \{M \in \text{GL}_n(E) : \sigma(M)^t = M^{-1}\} = \mathcal{G}(F)$

$\mathcal{G}(F)$ algebraic group defined over $F$

$\text{U}_n(E)$ will be different.
Main reference: Springer, Linear algebraic groups

Definition: The Lie algebra of a linear algebraic group \( G = G(F) \) is the tangent space of \( G \) at \( I \) in \( \text{Mat}_n(F) \).

If \( G = \{ M \in \text{Mat}_n(F) : f_i(M) = 0 \ \forall i \} \), then \( g = \{ M \in \text{Mat}_n(F) : \langle g M f_i, M \rangle = 0 \ \forall i \} \).

(more generally: \( g = \text{Der} \left( \mathcal{O}(G, \mathcal{L}) / \mathcal{O}(G, \mathcal{L})^2 \right) \)) regular forms vanishing at \( I \).

The adjoint representation of \( G \) on \( g \) is given by \( \text{Ad}_A(X) = AXA^{-1} \).

Example: \( G = \text{SL}_n(F) = \{ A \in \text{Mat}_n(F) : \det A = 1 \} \)

\( g = \text{sl}_n(F) = \{ A \in \text{Mat}_n(F) : \langle \det, A \rangle = 0 \} \)

\[ \frac{d}{dt} \det(A_t) = \sum_{ij} \frac{\partial \det(A)}{\partial x_{ij}} \frac{dx_{ij}}{dt} = \sum_{i=1}^n \frac{dx_{ii}}{dt} \]

\[ \Rightarrow \text{sl}_n(F) = \{ A \in \text{Mat}_n(F) : \langle \sum_{i=1}^n dx_{ii}, A \rangle \equiv \text{tr} A = 0 \} \]

Similar: \( \text{Lie} \left( \mathcal{O}_n(F) \right) = \mathcal{O}_n(F) = \{ M \in \text{Mat}_n(F) : M^t = -M \} \)

Exponential map: \( \exp : g \to G \)

\[ X \mapsto \sum_{n=0}^\infty \frac{1}{n!} X^n \quad \text{make sense of this!} \]

Need topology + division by \( n! \) \( F \) local field of characteristic \( 0 \).

Lemma: Let \( F \) be a finite extension of \( \mathbb{Q}_p \) and \( G \subset \text{GL}_n(F) \) an algebraic group. For \( X \in \mathfrak{p}^2 \mathcal{O}_n(\mathfrak{g}) \), \( \exp(X) \) is well-defined.

Proof: \( v_p(n!) = \sum_{m=1}^n v_p(m) \leq n \).

\[ \left| \frac{X^n}{h^n} \right|^p = \left| \frac{p^n}{h^n} \right|^p \leq p^{-n} \Rightarrow \sum_{n=0}^\infty \frac{X^n}{h^n} \text{ converges in } \mathcal{O}_n(F) \]

On \( \mathfrak{p}^2 \mathcal{O}_n(\mathfrak{g}) \), \( \exp \) has the usual properties, in particular

\[ \frac{d}{dt} \exp(tX) = X. \]
There is no exponential map over \( \mathbb{F}_q \langle t \rangle \).

Groups over \( \mathbb{Q}_p \) are somewhat easier than over \( \mathbb{F}_q \langle t \rangle \).

Example: \( \text{Func} \left( \frac{\text{SL}_2(F)}{\pm I_3} \right) = \frac{\text{Func} \left( \frac{\text{SL}_2(F)}{\pm I_3} \right)}{\text{PSL}_2(F)} = \frac{\text{Func} \left( \frac{\text{SL}_2(F)}{\pm I_3} \right)}{\text{SL}_2(F)} \)\\
\text{points of PSL}_2(F) = \text{maximal ideals of } F[x_1, x_2, x_3, x_4] \text{ even} / (x_1 x_4 - x_2 x_3 - 1) ;\\
\begin{itemize}
  \item \( \begin{pmatrix} A & \ast \\ \ast & A^t \end{pmatrix} : A \in \text{SL}_2(F) \)
  \item e.g. for \( F = \mathbb{Q}_p \) ( \( p \) not a square in \( \mathbb{Q}_p \) !)
\end{itemize}

\( B = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \frac{1}{\sqrt{p}} \end{pmatrix} \in \text{PSL}_2(F) \) because \( x_1 (B) = p \)
\( x_3 (B) = 0 \)
\( x_4 (B) = 1 \)

\( B \) defines an algebra homomorphism \( \text{Func} \left( \text{PSL}_2(F) \right) \to F = \mathbb{Q}_p \).

\( \Rightarrow \) The quotient map \( \text{SL}_2(F) \to \text{PSL}_2(F) \) is not surjective.

(Reason: \( \mathbb{Q}_p \) not algebraically closed.)

\( \square \)

Some kinds of linear algebraic groups

An algebraic group \( G \) is connected if the underlying variety is connected.

This does not imply that \( G(F) \) is connected w.r.t. the topology coming from \( F \).

Example: \( GL_n(F) \) is connected (can't separate \( \det \geq 0 \), using polynomials)\n\( \mathbb{O}_n(F) = \text{SO}_n(F) \cap \text{M}_n(F) ; \det H = 1 \) is disconnected.

\( G \) is simple if \( G, \{ 1 \} \) are the only (connected) normal algebraic subgroups of \( G \) (\( G \) should be connected and noncommutative).

Examples: \( SL_n, SO_n, Sp_{2n} \)

\( G \) is semisimple if \( G \) is an almost direct product of simple subgroups of \( G \).

This means that the product map \( \prod G_j \to G \) is surjective and has finite kernel.
Def: $\mathcal{G}$ is unipotent if it is isomorphic to an algebraic subgroup of $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

( so $G=1$ is nilpotent $\forall g \in G(F)$ )

Example: $G_2(F) = F = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \}$

Def: A linear algebraic group $\mathcal{G}$ is reductive if $\frac{\mathcal{G}}{\mathcal{Z}}$ is the only connected, normal, unipotent, algebraic subgroup of $\mathcal{G}$.

Examples: $Gln$, all semisimple groups.

Alternative characterization: $\mathcal{G}$ is reductive $\iff [\mathcal{G}, \mathcal{G}]$ is semisimple and $\mathcal{G}$ is the almost direct product of $[\mathcal{G}, \mathcal{G}]$ and the center $\mathcal{Z}(\mathcal{G})$.

From the representation theoretic point of view, reductive groups are the most interesting ones.

Def: An algebraic torus is a linear algebraic group which is diagonalizable (over $F$)

Examples: $(\mathbb{G}_m)^n$, $SO_2(F) = \{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : x, y \in F, x^2 + y^2 = 1 \}$

$SO_2(F) = \{ \begin{pmatrix} \sqrt{1-y^2} & y \\ y & \sqrt{1-y^2} \end{pmatrix} : x, y \in F, x^2 + y^2 = 1 \}$

$SO_2(F) = \{ \begin{pmatrix} x+\sqrt{1-y^2} & 0 \\ 0 & x-\sqrt{1-y^2} \end{pmatrix} : (x+\sqrt{1-y^2})(x-\sqrt{1-y^2}) = 1 \}$

$\Rightarrow SO_2(F) \hookrightarrow F^* \quad \text{isomorphism}$

$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \mapsto x+\sqrt{1-y^2}$

Def: A torus $\mathbb{Z}(F)$ is $F$-split if $\mathbb{Z}(F) \subset \mathbb{G}_m(F)$, i.e. $\mathbb{Z}(F) \hookrightarrow \mathbb{G}_m(F)$ as an isomorphism of algebraic groups over $F$. That means $\mathbb{Z}(F) \hookrightarrow \mathbb{G}_m(F)$, i.e. given by polynomials $w$ coefficients in $F$.

Example: $SO_2(F)$ is $F$-split iff char $F \neq 2$ and $\sqrt{1-F} \in F$. 
Def: An \( F \)-rational character of \( G \) is a homomorphism of algebraic groups 

\[ \chi: G \to G_m \] 

which is defined over \( F \).

\( \chi \) induces \( \bar{\chi}: \bar{G}(F) \to \bar{F}^* \).

The collection of \( F \)-rational characters of \( G \) is a group \( \bar{X}^*(G(F)) \) with pointwise multiplication of maps \( G(F) \to \bar{F}^* \).

Lemma: Every \( F \)-rational character of \( G_m(F) \) is of the form \( x \mapsto x^n \) for some \( n \in \mathbb{N} \).

Proof: Let \( \chi \in \bar{X}^*(F^*) \). It induces an algebraic homomorphism

\[ \chi^*: \text{Fun} \left( \bar{G}_m(F) \right) \to \text{Fun} \left( \bar{G}_m(F) \right), \quad \chi^*(t) \in F[S, S^{-1}] \text{ invertible} \]

\[ F[t, t^{-1}] \to F[S, S^{-1}] \]

so \( \chi(t) = c t^n \) for some \( c \in F^* \), \( n \in \mathbb{N} \).

\[ \Rightarrow \chi(yt) = cy^n \quad \forall y \in F^* \]

but \( \chi(y^2) = \chi(y) \chi(y) \), so \( c = 1 \).

Consequence: \( \bar{X}^*(G_m(F)) \cong \mathbb{Z} \).

\[ \bar{X}^*(G_m(F)) \cong \bar{X}^*(T \times T) \cong \bar{X}^*(T(F)) \cong \mathbb{Z}^{\dim T} \]

if \( T \) is \( F \)-split.

Maximal Subtori (of reductive groups);

Def: A subtorus of \( G(F) \) is an algebraic subgroup \( T(F) \) which is a torus

(and \( T(F) \) is defined over \( F \)).

A maximal \( F \)-split subtorus

is a subtorus which is \( F \)-split and is maximal for these properties.

Example: \( SO_3(\mathbb{R}) \): \[ \{ \left( \begin{array}{ccc} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{array} \right) : x^2 + y^2 = 1 \} \]

is a maximal \( F \)-split subtorus of \( SO_3(\mathbb{R}) \).
Thm (Borel-Tits: "Groupes réductifs"): Let $G$ be a connected linear algebraic group defined over $F$. Then any two maximal $F$-split subtori of $G$ are conjugate in $G(F)$.

From now on $G$ will be connected and reductive, defined over $F$. We fix a maximal $F$-split subtorus $S = S(F)$ in $G = G(F)$.

**Def:** $G$ is $F$-split if $S$ is a maximal torus in $G$.

**Examples:** $GL_n, SL_n$ are split for any $F$.
$SO_n$ is not $F$-split.

**Def:** The Weyl group $W(G,S) = N_G(S)/Z_G(S)$ (finite group) acts on $S$ by conjugation.

**Example:** $G = SL_n(F), S = \text{diagonal matrices in } G \cong Z_G(S)$.
$N_G(S) = \{ \text{monomial matrices in } G \}$
$W(G,S) \cong S_n$ (symmetric group)

**Roots of $(G,S)$:** $S$ acts on $\text{Lie}(G)$ via the adjoint representation of $G$.

$$\implies \text{Lie}(G) = \bigoplus_{\alpha \in \Pi^W(S)} V_{\alpha} \quad \text{where } V_{\alpha} = \{ X \in \text{Lie}(G) : A \alpha(A)X \forall A \in S \}$$

The sum is direct, because it is an algebraic action of a torus.

$\alpha \in \Pi^W(S) \setminus \Pi^F(S)$ are the roots of $(G,S)$, $R(G,S) = \text{set of roots}$.

**Example:** $G = SL_n(F), S = \text{diagonal matrices in } G \cong Z_G(S)$.
$V_{\alpha_{ij}} = \text{Lie}\left( \begin{array}{ccc} x_{ii} & \cdots & x_{ii} \\ \vdots & \ddots & \vdots \\ x_{nn} & \cdots & x_{nn} \end{array} \right) \quad \text{where } F \cong \text{Lie}(S)$.

**Theorem (Borel-Tits):** $V = Z(S)/Z(\mathfrak{g}) \otimes \mathbb{R}$ real vector space with $W(G,S)$-action.
We realize it w/ a $W(G,S)$-invariant inner product. Then $R(G,S)$ is an integral root system in $V$ w/ Weyl group $W(G,S)$.

If $G$ is $F$-split $\implies R(G,S)$ is reduced, i.e., $\text{Root}(R(G,S)) = \{ \alpha, -\alpha \}$.
Example: \( G \cdot U_2 \cdot \left( E/F, J \right) = \{ A \in \text{Gl}_2(E) : A^T J A = J \} \), \( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)

\( \text{Gal}(E/F) = \{ \text{id}_E, \sigma \} \), \( E/F \) Galois extension of degree \( 2 \), \( \sigma \) acts on \( E \) by \( \sigma(x) = x \sigma(x) \).

Diagonal matrices in \( G \): \( \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in E, (a, b) \in E \} \)

\( \sigma(a) = 1 = c \sigma(a), \sigma(b) = 1. \)

The \( b \)'s do not give anything \( F \)-split (for \( F = \mathbb{R}, b \sigma(b) = \sigma(1) = 1 \) compact).

\( S = \{ \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} : a \in E \} \) maximal \( F \)-split forms.

Claim: \( \text{Frob}(E) = \{ b \in E : b \sigma(b) = 1 \} \) not \( F \)-split.

\( \text{Proof}: \ E = F(\mu_2), \sigma(a + b \sqrt{2}) = a - b \sqrt{2} \Rightarrow \text{Frob}(E) = \{ (a, b) \in F^2 : (a + b \sqrt{2})(a - b \sqrt{2}) \}

\( \text{Frob}(E) = \{ (c, d) \in E^2 : cd = 1 \} = \text{GL}_1(E) \)

\( \text{Frob}(x)^{(c, d)} = \frac{c}{d}, \text{Frob}(x)^{(c, d)} = (2x + c)^n \in E \) \( F \)-valued \( x = 0, n = 0. \)

\( x = 0, n = 0. \)

For which \( x, y, z \in E \) does \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) lie in \( G = U_2 ? \)

\( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma(1) & 0 \\ 0 & \sigma(1) \end{pmatrix} = \begin{pmatrix} 1 & x \sigma(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)

\( \Rightarrow \ x = z \sigma(x), y + \sigma(y) + x \sigma(z), 2z \sigma(x) \sigma(y) = \sigma \Rightarrow \{ (x, y, z) \in E^3 : y + \sigma(y) + x \sigma(z) = 0 \}

\( \dim \text{U}_2 = 1, \dim \text{U}_4 = 3. \)
Theorem: Let \( x \in \mathcal{R}(G, S) \).

a) A canonical connected unipotent \( F \)-algebraic subgroup \( U_x \) is

\[
\text{Lie}(U_x) = \begin{cases} \mathfrak{V}_x & \text{if } x \not\in \mathcal{R}(G, S) \\ \mathfrak{V}_x + \mathfrak{V}_{2x} & \text{if } x \in \mathcal{R}(G, S) \end{cases}
\]

b) \( \mathbb{Z}_G(S) \) and the \( U_x, x \in \mathcal{R}(G, S) \) generate \( G \).

c) If \( G \) is \( F \)-split \( \Rightarrow U_x \cong \mathbb{F}^* = \mathbb{G}_m(F) \) (that it is \( t \)-simple is nontrivial), in the non-split case, the dimension of \( \mathfrak{V}_x \) can be large.

Parabolic subgroups: Let \( \Delta \) be a basis of \( \mathcal{R}(G, S) \Rightarrow \mathbb{R}^*+ \mathbb{R}^- \) positive/-negative roots.

For \( \Omega \Delta \), let \( P_{\Omega} \) be the group generated by \( \mathbb{Z}_G(S) \) and the \( U_x, x \in \mathcal{R}(G, S) \).

Def: The groups \( P_{\Omega} \) are called the \underline{standard parabolic subgroups}. A general parabolic subgroup \( P \) is conjugate to some \( P_{\Omega} \).

\( G/P \) is a complete variety. This characterizes the parabolic subgroups.

Example: \( G = SL_3(F) \), \( P_{\Omega} = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \right\} \), \( P_{111} \cong G \).

\( P_{x2} = \left\{ \begin{pmatrix} x * * \\ * x * \\ * * x \end{pmatrix} \right\} \)

\( P_{23} = \left\{ \begin{pmatrix} * x * \\ * * x \\ * * x \end{pmatrix} \right\} \)

In general, the minimal parabolic subgroup does not have to be solvable, (\( \text{"non-\text{quasi-split groups}"}\). Examples are much more complicated (\( \text{e.g.} \, \mathbb{Z}_G(S) \) not a torus).
Assumptions: * F local nonarchimedean field w/ discrete valuation \( v: F \rightarrow \mathbb{Z} \cup \{0\} \).
* \( G \) connected reductive algebraic group defined over \( F \). \( G = G(F) \).


Example: \( GL_3(F) \)

\[ U_{0,13} = \left\{ \left( \begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) : x, y, z \in F \right\} \]

For \( \text{ker} R \)

\[ U_{k,14} = \left\{ U_k(x) : v(x) \gg k \right\} \text{ compact open subgroup of } U_k. \]

\[ U_i \cup \text{ker} R = U_k, \quad \bigcap_{i < k} U_{0,14} = \{ 1 \}. \]

Maximal torus \( S \), \( S_r = \left\{ \left( \begin{array}{ccc} x_1 & \cdots & x_k \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{array} \right) \in S : v(x_i) = r \forall i \right\}, \quad r \in \mathbb{R}_{\geq 0} \)

\[ S_r = S, \quad r < 0. \]

\[ n_{0,13} = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) = U_{0,13} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \in N_G(S) \]

The image of \( n_0 \) in \( W(G,S) = N_G(S)/Z_G(S) \) is the reflection \( s_k \).

Theorem (Chevalley): Let \( G \) be \( F \)-split, there exist isomorphisms \( U_k : F \rightarrow U_k \)
for all \( k \in R(G,S) \) w/ properties a) and b).

Proof:

\[ U_{k} = U_k(\nu^{-k}[k,0]) \quad k \in R. \]

\[ U_{0} = S_r = \left\{ s \in S : v(X(S)) - 1 \gg r \forall X \in X(S) \right\}, \quad r \in \mathbb{R}_{\geq 0} \]

\[ s_r = s, \quad r < 0. \]

\[ n_w = U_k(1) U_{-k}(1) U_k(1) \]

a) For \( \alpha, \beta \in R(G,S) \cup \{0\} \), \( [U_{k,1}, U_{k,1}] \leq \left\langle \bigcup_{\alpha, \beta \in R(G,S) \cup \{0\}} \text{image generated by these tuples} \right\rangle \).

b) \( n_0 \in N_G(S) \), image of \( n_0 \) in \( W(G,S) \) is \( S_k \)

\[ n_0 \text{ in } W(G,S) \text{ is } S_k, \quad n_0 U_{-k}(x) n_0^{-1} = U_k(-x) \]

This is contained in Chevalley's proof that \( G \) can be defined over \( \mathbb{Z} \) (split, reduced).
Bruhat and Tits interpreted this as \( G \) has a prolonged valued root datum. This means in particular that \( U_\nu \) is filtered by compact open subgroups \( U_{\nu,r}, r \in \mathbb{R}^+ \), such that (a) holds. The assertion of (b) has to be refined in general.

Thin ("Bruhat + Tits," Groupes redéfinis en un corps local")

Every connected reductive pro-Fic group has a prolonged valued root datum.

This is what one needs to construct the affine building of \( G \).

**Example:** \( E/F \) Galois extension of degree 2, \( V \in \{ Y \in \mathbb{F} \mathcal{Y}(Y) \} / \mathcal{Y} \in \frac{1}{2} \mathbb{Z} \)

\[
G = U_3(E/F_0) \quad \mathcal{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
S = \left\{ \begin{pmatrix} a_x \ b_x \\ c_x \ d_x \end{pmatrix} : a \in F \right\} \quad U_{2x,1b} = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in E, y + \sigma(y) = 0, v(y) > 0 \right\}
\]

\[
U_{x,1b} = \left\{ \begin{pmatrix} x \ y \\ 0 \ 0 \end{pmatrix} : x, y \in E, x + \sigma(y) + x \sigma(x) = 0, v(x) > 2, v(y) > 2 \right\}
\]

The standard apartment of the affine building:

\[
\chi = \mathcal{X} (G/\mathcal{S}(G)) \quad A_S = \text{Hom}_\mathbb{Z} (X, \mathcal{R})
\]

For \( \nu \in \mathcal{R}(G, S) \) let \( \nu^* = \{ \nu \in \mathcal{R} : U_\nu \text{ jumps at } k \nu \} \). This is a discrete subgroup of \( \mathcal{R} \) containing \( \mathbb{Z} \). For \( \nu \in \mathcal{R}(G, S) \) \( \mathcal{P}_\nu = \mathcal{P}_{\nu^*} \).

**Def:** A wall in \( A_S \) is a hyperplane of the form \( H_{\alpha, \chi} = \{ \chi \in A_S : \langle Y, \alpha \rangle = \nu \} \), \( \alpha \in \mathcal{R}(G, S) \), \( k \nu \in \mathcal{P} \).

These make \( A_S \) into a polyhedral complex.

**Example:** For \( G = SL_3(F) \), \( \mathcal{R}(G, S) = \{ a \} \)

\[
\mathcal{S}_{aff} = \{ \mathcal{S}_{10}, \mathcal{S}_{10}, \mathcal{S}_{10} \}
\]
we fix a chamber $C_0$ with $x_0 \in C_0$. $S_{1,k} = \text{affine reflection in } H_{k_1}$.

$W_{aff} := \text{subgroup of } A_5 \ltimes W(G,s) \text{ generated by } S_{1,k}, (k \in P^1)$

$\text{Suff} := \{ S_{1,k} : H_{k_1} \text{ is a wall of } C_0 \}$

Thus (Bourbaki):

a) $(W_{aff}, \text{Suff})$ is a Coxeter system.

b) $A_5$ is the associated Coxeter complex, i.e.,
   1) $W_{aff}$ acts simply transitively on the set of chambers.
   2) The neighbors of $WC_0$ are $\{ WC_0 : s \in \text{Suff} \}$.

The action of $N_G(s)$ on $A_5$:

Define $\nu : \mathbb{Z}_G(s) \to A_5$, $\langle \nu(s), x \rangle := -\nu(x(s)), x \in X^G(G(s)/N_G(s))$

In the above example for $s = (p, v, i), \nu(p) = 1 \to \nu(x(s)) = \nu(p^2) = 2$

$v(x(s)) = \nu(p^2) = 1$

$v(s) = -\alpha^v := -\frac{2}{\langle \alpha, \nu \rangle}$

$v$ can be extended to $\nu : \mathbb{Z}_G(s) \to A_5$ and further to $\nu : N_G(s) \to A_5 \ltimes W(G,s)$

such that $\nu$ induces $\text{id}(W(G,s)) : N_G(s)/\mathbb{Z}_G(s) \to A_5 \ltimes W(G,s)$

ker $\nu = \mathbb{Z}(G) \cdot (\text{maximal compact subgroup of } \mathbb{Z}_G(s))$

for $G$ split: $\nu(u) = S_{w,0} (\iff \nu(w(x)) u_{w,0}(x) = u_{w,0}(x))$

This determines $\nu$ uniquely (in the split case).

In the example,

$g = U_{w_0}(x) U_{-w_0}(x) U_{w_0}(x) = (0, x, 0) = (x, 0, x) = S_{w,0}

v(x) = \nu(x, x, 1, 0) S_{w,0} = (x, x, 1, 0) \in A_5 \ltimes W(G,s)$

$A_5 \ltimes W(G,s)$ acts on $A_5$ by $(x, y) \cdot g = x + w(y)$, so $\nu$ defines an action of $N_G(s)$ on $A_5$. $B(G)$ will be $G \ltimes A_5/\mathbb{Z}$.

[Reference: Tits, Corvallis proceedings, "Reductive groups over a local field"]

We need isotropy groups (in $G$) of points of $A_5$. We decree that

the fixed points of $U_{w,0}$, $w \in P^1$, are $\{ y \in A_5 : w(y) \}$, a half space

in $A_5$. 


\[ G_y = \text{group generated by } N_\sigma(s)_y \text{ and } \bigcup_{\alpha \in \mathbb{R}(G_s)} U_{\alpha,-\alpha(y)} \quad , \quad y \in A_1, \]

\[ G_y \text{ is not larger than expected : } G_y \cap U_\alpha = U_{\alpha,-\alpha(y)}. \]

\[ G_y / Z(G) \cap G_y \text{ is compact.} \]

Borel-Tits : \[ G_y = (U^- \cap G_y)(N_\sigma(s) \cap G_y)(U^+ \cap G_y) \] (as sets)

where \( U^\pm \) are the groups generated by \( \bigcup_{\alpha \in \mathbb{R}(G_s)} U_{\alpha} \) (for any reasonable choice of positive roots).

**Example:** \[ SL_3(\mathbb{O}_p)_0 = SL_3(\mathbb{Z}_p) \]

\[ SL_3(\mathbb{O}_p)_y = \left( \begin{array}{ccc} \mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p & \mathbb{Z}_p \\ p^2\mathbb{Z}_p & p\mathbb{Z}_p & \mathbb{Z}_p \end{array} \right) \cap SL_3(\mathbb{O}_p) \] "Iwahori subgroup" \[ y \in G_y \text{ interior point.} \]
Def: The Bruhat-Tits building of $G = G(F)$ is $G \times \mathbb{A}^1 / \mathbb{A}^1$, where $(g, x) \sim (h, y)$ iff $\exists n \in \mathbb{N}_0(s), n(x) = y$, i.e., $g^{-1}h \in G_x$.

Example: $G = SL_2(\mathbb{Q}_3)$

\[ G_0 = SL_2(\mathbb{Z}_3), \quad G_x = \left( \frac{\mathbb{Z}_3 \times \mathbb{Z}_3}{\mathbb{Z}_3 \times \mathbb{Z}_3} \right) \cap SL_2(\mathbb{Q}_3) \]

\[ G/G_x \cong U_{-x, 0} / U_{-x, 1} \cong \mathbb{Z}_3 / 3\mathbb{Z}_3 \]

Properties:

1) $\mathbb{A}^1$ embeds in $B(G)$ via $g \mapsto (1, g)$, (easy!)

2) $G$ acts on $B(G)$ by $g \cdot (h, y) = (gh, y)$

3) The isotropy group of $x \in \mathbb{A}^1$ is indeed $G_x$.

4) $x y \in \mathbb{A}^1$ in one $G$-orbit $\Rightarrow x / y$ in the same $\mathbb{N}_0(s)$-orbit.

5) $Z(G)$ acts trivially on $B(G)$, because $Z(G) \subseteq G_x \forall x \in \mathbb{A}^1$.

6) terminology: $g \mathbb{A}^1$ apartment, $g \mathbb{C}$ chamber ($\mathbb{C}$ chamber in $\mathbb{A}^1$), $g \mathbb{S}$ poly-simplex in $B(G)$ ($\mathbb{S}$ poly-simplex in $\mathbb{A}^1$)

7) $B(G)$ polysimplicial complex, locally finite

8) The action of $G$ preserves the polysimplicial structure.
9.) For any two polysimplices \( \sigma, \tau \) in \( B(G) \), there is an apartment containing \( \sigma, \tau \).

In the picture: apartments = lines extending indefinitely in both directions

\[ G\gamma = \{ \gamma g \in G : g \gamma = \gamma \forall \gamma g \in Y \} \]

**Lemma:**

a) Let \( \sigma \) be a polysimplex in \( B(G) \) \( \Rightarrow \) \( G_\sigma \) acts transitively on the set of apartments containing \( \sigma \).

b) Suppose \( \sigma \subset A \Rightarrow G_\sigma A = B(G) \). (Any apartment)

**Proof:**

a) We may assume that \( \sigma \subset A_0 \). For generic \( k \in G \), \( G_k = G_\sigma \) because \( G \) respects polysimplices. Suppose that an apartment \( A \) contains \( \sigma \), \( A = gA_0 \) for some \( g \in G \). Since \( g^{-1}x \in A_0 \)

\[ \exists n, m : g^{-1}x = nx \quad \text{i.e.,} \quad g_{nx} = x \Rightarrow g_{nx}G_k, \]

\[ A = gA_0 = gA_k. \]

b) Follows from a) and b).

To understand the relation between \( B(G) \) and the picture visualizing it, observe that e.g.,

\[ \frac{SL_2(\mathbb{Z}_3)}{A_5} = \frac{SL_2(\mathbb{Z}_3)}{SL_2(\mathbb{Z}_3) \cap G_{x_4}} \cong \frac{U_{-n,0}}{U_{-n,4}} \]

\[ \left( \frac{\mathbb{Z}_3}{\mathbb{Z}_3}, \frac{3^{-n}\mathbb{Z}_3}{\mathbb{Z}_3} \right) \cong SL_2(\mathbb{Q}_3) \]

For \( G = SL_2(F) \), \( A_5 = \) along every wall \( H \) of \( A_5 \), \( B(SL_3,F) \) branches, the branches are parametrized by

\[ \frac{U_{n,0}/U_{n,1}U_{n,0}}{U_{n,0}/U_{m,0}} \cong \frac{SL_2(\mathbb{Q})}{T_8(\mathbb{Q})} \]

\( \sigma = \exists x \in F : \sigma(x) \) is fixed, \( \iff \) it fixes a point of \( B(G) \).
Corollary: There is a bijection \( \{ \text{Vertices of } B(G) \} \leftrightarrow \{ \text{maximal compact subgroups of } G \} \) \nolimits \nulldelimiterspace=0pt
\begin{align*}
\begin{array}{c}
K_x = \text{maximal compact subgroup of } G_x \\
\mathbb{G}_x \cap [\mathbb{E}, \mathbb{E}] \mathbb{Z}_{\mathbb{E}}
\end{array}
\end{align*}

For \( G = \text{SL}_3(\mathbb{F}) \) as above:
\( K_0 = \text{SL}_3(\mathbb{O}) \)
\( K_x = \begin{pmatrix}
\sigma & 0 & 0 \\
0 & \sigma & 0 \\
0 & 0 & \sigma
\end{pmatrix} \cap \text{SL}_3(\mathbb{F}) \), not conjugate to \( K_0 \).
\[ \{ K_0, K_x, K_y \} \] all conjugacy classes.

**Def:** A vertex \( x \in \mathbb{G}_c \) is **special** if \( N_{\mathbb{G}}(s)_x / \mathbb{Z}_{\mathbb{G}}(s)_x \cong \mathbb{W}(G, s) \).

\( \mathbb{G} \in \mathbb{G}_c \) is special.

**Example:**

For \( \text{SL}_3(\mathbb{F}) \), all vertices are special.

Fix a basis \( \Delta \) of \( R(G, s) \).
Positive cone \( A_3^+ = \{ x \in \mathbb{A}_3 : \langle x, x \rangle > 0 \} \).
\[ \nu : N_{\mathbb{G}}(s) \rightarrow A_3 \times \mathbb{W}(G, s) \] \begin{align*}
\mathbb{Z}_{\mathbb{G}}(s)^+ & = \nu^{-1}(A_3^+) \nulldelimiterspace=0pt
\end{align*}

**Theorem** (Cartan decomposition)

a) Let \( x \in \mathbb{G}_c \) special vertex \( \Rightarrow G = K_x \mathbb{Z}_{\mathbb{G}}(s)^+ K_x 
\)
b) The natural map \( \mathbb{Z}_{\mathbb{G}}(s)^+ / \ker \nu \rightarrow K_x / G / K_x \) is bijective.

**Proof:** a) Let \( g \in G, x \in B(G) \). By Lemma 7 \( \exists k \in K_x \text{ s.t. } k g k^{-1} x \in \mathbb{A}_3 \)
\[ \Rightarrow \exists e \in N_{\mathbb{G}}(s) : e(x) = y, g^{-1} k^{-1} e \in G_x \text{ \text{young} } A_3^+ \]
There exists \( t \in A_s^+ \cap \nu(Z_6(5)) \), \( w \in \frac{N_6(5)}{Z_6(5)} \), s.t. \( y = w(\nu t) \). Pick \( z \in Z_6(5) \).

Since \( G_X \in \mathcal{Z}(6) \) and \( \mathcal{Z}(6) \in \mathcal{Z}(S)^+ \), we can achieve \( g \in K_X Z_6(5) \).

b) Suppose that \( g \in K_X Z_6 \cap K_X Z_6' K_X \). Show \( z'z'' \in \ker(v) \).

If \( x \in K_X \), then \( K_X (zX) \cap K_X (z'X) \Rightarrow (zX) = K_X (z'X) \).

\[ zX = z'X \Rightarrow z \ker(v) = z' \ker(v) \] (\( \nu(z) = \nu(z') \)).

We have \( N_6(5) / \ker(v) \), a finite extension \( W \).

Let \( C \) be a chamber in \( A_s^+ \).

Thus, (affine Bruhat decomposition) Suppose \( Z(6) \) is compact.

a) \( G = G_c N_6(5) G_c \approx G_c W G_c \) (simplified written)

b) \( W \rightarrow G_c \backslash G / G_c \) bijection.

Proof: Like for the Cartan decomposition.

Example: \( G = \text{SL}_n(F) \), \( G_c = \left( \begin{array} \otimes \end{array} \right) \cap \text{SL}_n(F) \).

There's also an Iwahori decomposition. The main point in the proof is to show that \( U^+ A_s = B(G) \).
Substantial fragment of a two-dimensional building

This emphasizes the visually chaotic nature of any two-dimensional representation of a thick building of dimension greater than one. One-dimensional affine buildings are simply texts, so can be rendered in a comprehensible and illuminating (as well as aesthetically interesting) manner. But in higher dimensions the thickness of the building is a very direct obstacle to creation of accurate two-dimensional models.
\[ U^+ := \text{group generated by } U^+ U_\infty. \]

Lemma 2 \( U^+ \cdot A_+ = B(G) \).

Proof. Let \( y \in B(G) \).

Choose a chamber \( C \subset A_+ \) "sufficiently" deep inside \( A_+ \).

Fix \( x \in A_+ \) such that \( \forall y \in G_c \cdot x \) and \( x + A_+ \supset C \).

by Lemma 1.b.

\[
\begin{align*}
g \in G_c &= (G_c \cap U^+) \cdot (G_c \cap N_c(s)) \cdot (G_c \cap U^-) \\
&= \ker v \\
&= \text{ker } v \\
&\text{since } C \subset A_+ \text{ is open} \\
&\text{fixes } x \text{ since } v \text{ fixes } x \\
&\forall x \in \mathbb{R}^2(s) \cap C \\
&-\alpha(x) \leq -\alpha(a) \\
&U_{a_\alpha}(x) \supset U_{a_{-\alpha}(a)} \\
&\text{for suitable } a_\alpha \text{ for } \alpha \\
&U_{a_\alpha} \cap G_x \supset U_{a} \cap G_c
\end{align*}
\]

So \( y = g \cdot x = u \cdot x \) for some \( u \in G_c \cap U^+ \). \( \square \)

Theorem \( (\text{Iwasawa decomposition}) \)

Let \( x \in A_+ \) be special and let \( P_0 = U^+ Z_G(s) \) be the standard minimal parabolic subgroup of \( G \).

a) \( G = P_0 \cdot K \cdot P_0 \)

b) For any parabolic \( P \) and any good maximal compact subgroup \( K \) : \( G = P K = KP \).

Proof. a) Let \( g \in G \). By Lemma 2, \( \exists u \in U^+, y \in A_+ \) such that \( g \cdot x = u \cdot y \)

\[ \Rightarrow \exists h \in N_G(s) : h(x) = y, \quad g^{-1} u h \in G_x \]

Since \( x \) is special, we can write \( u \cdot h^{-1} = z \cdot k \) with \( z \in Z_G(s) \) and \( k \in K_x \) \( \Rightarrow y = u \cdot h = u \cdot z \cdot k \in P_0 \cdot K \).
b) By definition \( \exists g \in G : g P g^{-1} = P \).

By a) we can write \( g = p k \) with \( p \in P, k \in K \).

\[ PK_x \supseteq g^{-1} P g K_x = k^{-1} p^{-1} P p k K_x = k^{-1} P k x = k^{-1} G \]

\[ \implies PK_x = G = k x P \]

Moreover, \( k \) is conjugate to some \( k_y \) with \( y \in A_y \) as special.

\[ k = h k_y h^{-1} \implies PK = P h k_y h^{-1} = h (h^{-1} P h) k_y h^{-1} = h G h^{-1} = G, \quad \Box \]