Strong pure infiniteness of crossed products

joint work with Eberhard Kirchberg

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Question

Why study strongly purely infinite C-algebras?*

Theorem

Let A, B be nuclear, separable, stable C-algebras both with primitive ideal space isomorphic to a T_0-space X. Then*

(i) $A \otimes \mathcal{O}_\infty \cong B \otimes \mathcal{O}_\infty \Leftrightarrow A$ *and* B *are* KK_X*-equivalent.*

(ii) $A \otimes \mathcal{O}_\infty \cong A \Leftrightarrow A$ *is strongly purely infinite (s.p.i.).*

- KK_X-equivalence $=$ KK-theory that respects the primitive ideal spaces.
- A s.p.i. $=$ for every $\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)_+$ and $\varepsilon > 0$ there exist $d_1, d_2 \in A$, such that $\left\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| < \varepsilon$.

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Theorem

Let A be a C^*-algebra. Then A is s.p.i. iff $F := A_+$ has the diagonalization property.

- $F \subseteq A_+$ has the *diagonalization property* = for every matrix $[a_{jk}] \in M_n(A)_+$, $a_{jj} \in F$, $n \in \mathbb{N}$ and $0 < \tau < \varepsilon < 1$ there exist $d_1, \ldots, d_n \in A$, s.t.

$$d_j^* a_{jj} d_j = (a_{jj} - \varepsilon)_+, \quad \|d_j^* a_{jk} d_k\| < \tau, \quad j \neq k.$$

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Theorem

Let A be a C-algebra and $F \subseteq A_+$. If*

(i) *F has the diagonalization property.*

(ii) *F is a filling family.*

Then A is s.p.i.

- $F \subseteq A_+$ is a *filling family* = for every hereditary $D \subseteq A$ and ideal $I \subseteq A$ with $D \not\subseteq I$ there exist $z \in A$ s.t.

$$z^* z \in D \not\subseteq I, \quad zz^* \in F.$$

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Remark

For crossed products we have a natural choice for F consisting of the positive elements in the original algebra.

$$F := A_+ \subseteq (A \rtimes_r G)_+$$

Definition

- G = discrete group
- A = C*-algebra in $B(H)$
- action = automorphism $a \mapsto t.a$ ($t \in G$, $a \in A$)
- $A \rtimes_r G$ = the norm closure of the image of the induced map $\pi\colon C_c(G, A) \to B(\oplus_{t \in G} H)$ defined by

$$a \mapsto [a_{t,s}], \quad a_{t,s} = t^{-1}.a(ts^{-1})$$

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Question

When is $F := A_+$ *a filling family for* $(A \rtimes_r G)_+$?

Theorem

Let G be a discrete group acting by an exact action on a separable C-algebra A. Suppose that the action on any quotient A/I by a G-invariant closed ideal $I \neq A$ is elementwise properly outer. Then $F := A_+$ is a filling family for $(A \rtimes_r G)_+$.*

- exact = every G-invariant ideal $I \subseteq A$ induces a short exact sequence at the level of reduced crossed products
- properly outer automorphism α of A = for every α-invariant ideal $0 \neq I \subseteq A$ and inner automorphism β of I, $\|\alpha|_I - \beta\| = 2$.
- elementwise properly outer action = for evert $t \neq e$, $t \mapsto t.a$ is property outer

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Example

- $G = \mathbb{Z}$
- $A = M_{n^\infty} \otimes \mathcal{K}$
- action = shift (from $a \mapsto e_{11} \otimes a$)

$F := A_+$ is a filling family for $(A \rtimes_r G)_+$

Example

- $G = \mathbb{Z}_2 * \mathbb{Z}_3$, $*$ = free product
- $A = C(\partial G)$, ∂G = infinite word space
- action = word concatenation

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Question

When does $F := A_+$ *in* $(A \rtimes_r G)_+$ *have the diagonalization property?*

Theorem

Let G be a discrete group action on a C-algebra A. Suppose that the action is G-separating. Then* $F := A_+ \subseteq (A \rtimes_r G)_+$ *has the diagonalization property.*

- G-separating action on A = for every $a, b \in A_+$, $c \in A$, $\varepsilon > 0$, there exist $s, t \in A$ and $g, h \in G$ s.t

$$\|s^*as - g.a\| < \varepsilon, \ \|t^*bt - h.b\| < \varepsilon, \ \|s^*ct\| < \varepsilon.$$

Remark

An action on a unital abelian C*-algebra is never G-separating.

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An action on a unital abelian C*-algebra is never G-separating.

Corollary

Let G be a discrete group acting on a separable C-algebra A. Suppose that the action of G on A is exact, G-separating, and the action on any quotient A/I by a G-invariant ideal $I \neq A$ is elementwise properly outer. Then $A \rtimes_{\sigma,r} G$ is s.p.i.*

Example

Let G be a discrete group acting by an exact, essentially free action on an abelian C*-algebra $C_0(X)$. Suppose that the action is G-separating, i.e., for every $U_1, U_2 \subseteq X$ open and $K_1, K_2 \subseteq X$ compact, with $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, there exist $g, h \in G$ s.t.

$$g.K_1 \subseteq U_1, \quad h.K_2 \subseteq U_2, \quad g.K_1 \cap h.K_2 = \emptyset.$$

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Suppose that φ is an endomorphism of a separable C*-algebra A. Let $(A_\infty, \varphi_m \colon A \to A_\infty)$, be the inductive limit of the sequence

$$A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$$

φ induces an automorphism of A_∞. If

- the action on any quotient A_∞/I by a G-invariant ideal $I \neq A_\infty$ is elementwise properly outer
- for every $a, b \in A_+$, $c \in A$ and $\varepsilon > 0$ there exist $k \in \mathbb{N} \cup \{0\}$ and elements $e_1, e_2 \in A$ such that

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- $A = M_{n^\infty} \otimes \mathcal{K}$
- $G = \mathbb{Z}$
- action = shift (from $a \mapsto e_{11} \otimes a$)

The action is G-separating. Hence $A \rtimes_r G$ ($\cong \mathcal{O}_n \otimes \mathcal{K}$) is s.p.i.

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Proof.

Enough if for $a, b, c \in M_{n^m}$ there exist $s, t \in M_{n^{m+1}}$ s.t.

$$s^*as = 1.a, \;\; t^*bt = 1.b, \;\; s^*ct = 0$$

With the inclusion $M_{n^m} \subseteq M_{n^{m+1}}$, $a \mapsto a \otimes 1_n$ this is just

$$s^*(a \otimes 1_n)s = e_{11} \otimes a, \quad t^*(b \otimes 1_n)t = e_{11} \otimes b,$$
$$s^*(c \otimes 1_n)t = 0;$$

easily obtained using elementary matrix operations. □

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$$s^*(a \otimes 1_n)s = e_{11} \otimes a, \quad t^*(b \otimes 1_n)t = e_{11} \otimes b,$$
$$s^*(c \otimes 1_n)t = 0;$$

easily obtained using elementary matrix operations. □

Proof.

Enough if for $a, b, c \in M_{n^m}$ there exist $s, t \in M_{n^{m+1}}$ s.t.

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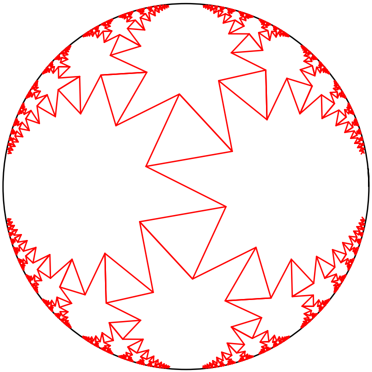

Figure: The Cayley graph for $\mathbb{Z}_2 * \mathbb{Z}_3$ with all edges of unit length.

Thank you for your attention :o)