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Definition

For $(A_i) \subseteq (A, \omega)$ unital C*-subalgebras we say (A_i) are free in A if

$$\forall n \in \mathbb{N} \quad \forall a_j \in \mathcal{A}_{i_j}, 1 \leq j \leq n, i_j \neq i_{j+1}, \omega(a_j) = 0$$

we have

$$\omega(a_1a_2\cdots a_n)=0.$$

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B(H)Equivalence $\Phi_{x,y}^{(i)}$

Open questions Given unital C*-algebras (A_i, ω_i) construct algebra (A, ω) such that

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• (\mathcal{A}_i) free in (\mathcal{A}, ω)

•
$$\omega|_{\mathcal{A}_i} = \omega_i$$

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Open questions Given unital C*-algebras (A_i, ω_i) construct algebra (A, ω) such that

• (\mathcal{A}_i) free in (\mathcal{A}, ω)

•
$$\omega|_{\mathcal{A}_i} = \omega_i$$

Notation

$$(\mathcal{A},\omega) = *_i(\mathcal{A}_i,\omega_i)$$

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Open questions • $\forall i : A_i \subseteq B(H_i) \Rightarrow A \text{ acts on } *_i H_i$

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B(H) Equivalence $\Phi_{X,V}^{(i)}$

Open questions $\forall i : \mathcal{A}_i \subseteq B(H_i) \Rightarrow \mathcal{A} \text{ acts on } *_i H_i$ $*_i C_r^*(G_i) = C_r^*(*_i G_i)$

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Open questions ■ $\forall i : A_i \subseteq B(H_i) \Rightarrow A \text{ acts on } *_i H_i$ ■ $*_i C_r^*(G_i) = C_r^*(*_i G_i)$ ■ $*_i L(G_i) = L(*_i G_i)$

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Open questions

- $\forall i : A_i \subseteq B(H_i) \Rightarrow A \text{ acts on } *_i H_i$
- $\bullet *_i C_r^*(G_i) = C_r^*(*_i G_i)$
- $*_i L(G_i) = L(*_i G_i)$
- Dense subset $A = \mathbb{C}1 \oplus \bigoplus_n \bigoplus_{i_1,\dots,i_n,i_j \neq i_{j+1}} \mathring{\mathcal{A}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{A}}_{i_n}$ where $\mathring{\mathcal{A}}_i = \ker \omega_i$

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B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_{\phi}(a_1\ldots a_n)=\phi(n)a_1\ldots a_n.$$

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Open questions Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(\mathsf{a}_1\dots\mathsf{a}_n)=\phi(n)\mathsf{a}_1\dots\mathsf{a}_n.$$

• Is M_{ϕ} welldefined on \mathcal{A} ?

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Open questions Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(a_1\ldots a_n)=\phi(n)a_1\ldots a_n.$$

- Is M_{ϕ} welldefined on \mathcal{A} ?
- When is M_{ϕ} completely bounded?

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Open questions Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(a_1\ldots a_n)=\phi(n)a_1\ldots a_n.$$

- Is M_{ϕ} welldefined on \mathcal{A} ?
- When is M_{ϕ} completely bounded?
- For which A_i ?

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Open questions Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(a_1\ldots a_n)=\phi(n)a_1\ldots a_n.$$

- Is M_{ϕ} welldefined on \mathcal{A} ?
- When is M_{ϕ} completely bounded?
- For which A_i ?
- For which ϕ ?

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Open questions Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_\phi(a_1\ldots a_n)=\phi(n)a_1\ldots a_n.$$

- Is M_{ϕ} welldefined on \mathcal{A} ?
- When is M_{ϕ} completely bounded?
- For which A_i ?
- For which ϕ ?
- $\blacksquare \|M_{\phi}\|_{cb} = ?$

$\mathsf{Class}\ \mathcal{C}$

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Equivalence $\Phi_{x,y}^{(i)}$

Open questions

Definition

Let $\mathcal C$ denote the set of functions $\phi:\mathbb N_0\to\mathbb C$ for which the Hankel matrices

$$h = (\phi(i+j) - \phi(i+j+1))_{i,j \ge 0}$$

$$k = (\phi(i+j+1) - \phi(i+j+2))_{i,j \ge 0}$$

are of trace class and $c = \lim_{n \to \infty} \phi(n)$ exists. For $\phi \in C$ put

 $\|\phi\|_{\mathcal{C}} = \|h\|_1 + \|k\|_1 + |c|.$

Known results

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Open questions

Theorem (Wysoczanski 1995)

Let $G = *_{i \in I} G_i$ and $\phi \in C$ then $M_\phi : C^*_r(G) \to C^*_r(G)$ is welldefined and

 $\|M_{\phi}\|_{cb} \leq \|\phi\|_{\mathcal{C}}.$

Theorem (Ricard-Xu 2006)

Let $\mathcal{A} = *_i \mathcal{A}_i$ and $\phi(n) = s^n$, $s \in (0, 1)$ then $M_{\phi} : \mathcal{A} \to \mathcal{A}$ is welldefined and

 $\|M_{\phi}\|_{cb} \leq 1.$

Our result

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Theorem (Haagerup-M 2010)

Let $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ be the reduced free product of unital C^* -algebras $(\mathcal{A}_i)_{i \in I}$ with respect to states $(\omega_i)_{i \in I}$ for which the GNS-representation π_{ω_i} is faithful for all $i \in I$. If $\phi \in C$, then there is an unique linear completely bounded map

 $M_{\phi}:\mathcal{A}
ightarrow\mathcal{A}$

such that $M_{\phi}(1) = \phi(0)1$ and

 $M_{\phi}(a_1a_2\ldots a_n)=\phi(n)a_1a_2\ldots a_n$

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whenever $a_j \in \mathring{A}_{i_j} = ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. Moreover $\|M_{\phi}\|_{cb} \leq \|\phi\|_{\mathcal{C}}$.

Example

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Sketch of proof B(H) Equivalence $\Phi_{ij}^{(i)}$

Open questions

Let
$$\mathbb{D}=\{s\in\mathbb{C}||s|<1\}.$$
 For every $s\in\mathbb{D}$ $\phi_s(n)=s^n$

defines a radial multiplier M_{ϕ} on $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ with

$$\|M_{\phi_s}\|_{cb} \leq \frac{|1-s|}{1-|s|}.$$

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Reduction to B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions

- Uniqueness of M_{ϕ}
- Reduce to $A_i = B(H_i)$
- Equivalent description of M_{ϕ}

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- Construct $\Phi_{x,y}^i$
- Construct T_1, T_2, T
- Show T is M_{ϕ}
- Estimate ||T||_{cb}

Reduce to $A_i = B(H_i)$

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Reduction to B(H) Equivalence $\Phi_{X+Y}^{(i)}$

Open questions • Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$

Reduce to $A_i = B(H_i)$

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Reduction to B(H) Equivalence $\Phi_{X,Y}^{(i)}$

Open questions • Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$

• Let $(A, \omega) = *_i(A_i, \omega_i)$

Reduce to $A_i = B(H_i)$

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Reduction to B(H) Equivalence $\Phi_{X,V}^{(i)}$

Open questions

- Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$
- Let $(A, \omega) = *_i(A_i, \omega_i)$
- Let (H_i, Ω_i) = (H_{ωi}, ξ_{ωi}) from GNS-representation of (A_i, ω_i) Denote by (H, Ω) = *_i(H_i, Ω_i)

Reduce to $A_i = B(H_i)$

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Reduction to B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions

- Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$
- Let $(A, \omega) = *_i(A_i, \omega_i)$
- Let (H_i, Ω_i) = (H_{ωi}, ξ_{ωi}) from GNS-representation of (A_i, ω_i) Denote by (H, Ω) = *_i(H_i, Ω_i)
- Now (A_i, ω_i) can be realized as subalgebra of $B(H, \Omega)$

Reduce to $A_i = B(H_i)$

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Reduction to B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions

- Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$
- Let $(A, \omega) = *_i(A_i, \omega_i)$
- Let (H_i, Ω_i) = (H_{ωi}, ξ_{ωi}) from GNS-representation of (A_i, ω_i) Denote by (H, Ω) = *_i(H_i, Ω_i)
- Now (A_i, ω_i) can be realized as subalgebra of $B(H, \Omega)$

• Use the theorem to find $M_{\phi}: B(H) \rightarrow B(H)$

Reduce to $A_i = B(H_i)$

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Reduction to B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions

- Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$
- Let $(A, \omega) = *_i(A_i, \omega_i)$
- Let (H_i, Ω_i) = (H_{ωi}, ξ_{ωi}) from GNS-representation of (A_i, ω_i) Denote by (H, Ω) = *_i(H_i, Ω_i)
- Now (A_i, ω_i) can be realized as subalgebra of $B(H, \Omega)$

- Use the theorem to find $M_{\phi}: B(H) \rightarrow B(H)$
- Then $M_{\phi}|_A : A \rightarrow A$ with right behaviour

Reduce to $A_i = B(H_i)$

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Reduction to B(H)Equivalence $\Phi_{x,y}^{(i)}$

Open questions

- Assume the theorem holds for $(B(H_i), \omega_{\Omega_i})$
- Let $(A, \omega) = *_i(A_i, \omega_i)$
- Let $(H_i, \Omega_i) = (H_{\omega_i}, \xi_{\omega_i})$ from GNS-representation of (A_i, ω_i) Denote by $(H, \Omega) = *_i(H_i, \Omega_i)$
- Now (A_i, ω_i) can be realized as subalgebra of $B(H, \Omega)$

- Use the theorem to find $M_{\phi}: B(H) \rightarrow B(H)$
- Then $M_{\phi}|_{A}: A \rightarrow A$ with right behaviour
- $||M_{\phi}|_{\mathcal{A}}||_{cb} \leq ||M_{\phi}||_{cb} \leq ||\phi||_{\mathcal{C}}$

Notation

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Reduction to B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions

$H = \mathbb{C}\Omega \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}.$

and denote basis by

$$\Lambda = \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{\gamma_1 \otimes \cdots \otimes \gamma_n | \gamma_j \in \mathring{\Gamma}_{i_j}, i_1 \neq \cdots \neq i_n\}.$$

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Reduction to B(H)Equivalence $\Phi_{x,y}^{(i)}$

Open questions

$H = \mathbb{C}\Omega \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}.$

and denote basis by

$$\Lambda = \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{\gamma_1 \otimes \cdots \otimes \gamma_n | \gamma_j \in \mathring{\Gamma}_{i_j}, i_1 \neq \cdots \neq i_n\}.$$

• For $\gamma \in H$, define $L_{\gamma} \in B(H)$ as

$$L_{\gamma}(\chi) = \begin{cases} \gamma \otimes \chi & \text{if } i \neq i_1 \\ 0 & \text{if } i = i_1 \end{cases}$$

For $\eta, \xi \in H$ let case 2 if $\eta_{|\eta|}, \xi_{|\xi|} \in H_i$ and case 1 otherwise.

Equivalent description of M_{ϕ}

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Lemma

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Reduction to B(H) Equivalence

 $\Phi_{X,Y}^{(i)}$

Open questions Let $T : B(H) \to B(H)$ be a bounded linear normal map, and let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function on \mathbb{N}_0 . TFAE

Equivalent description of M_{ϕ}

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Equivalence

Open questions Let $T : B(H) \to B(H)$ be a bounded linear normal map, and let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function on \mathbb{N}_0 . TFAE (a) $T(1) = \phi(0)1$ and

$$T(a_1a_2\ldots a_n)=\phi(n)a_1a_2\ldots a_n$$

whenever $a_j \in B(\mathring{H}_{i_j}) = ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$.

Equivalent description of M_{ϕ}

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Lemma

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Reduction to B(H)Equivalence $\Phi(i)$

Open questions Let $T : B(H) \to B(H)$ be a bounded linear normal map, and let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function on \mathbb{N}_0 . TFAE (a) $T(1) = \phi(0)1$ and

$$T(\mathsf{a}_1\mathsf{a}_2\ldots\mathsf{a}_n)=\phi(n)\mathsf{a}_1\mathsf{a}_2\ldots\mathsf{a}_n$$

whenever $a_j \in B(\mathring{H}_{i_j}) = ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. (b) For all $k, l \in \mathbb{N}_0$ and $\xi \in \Lambda(k), \eta \in \Lambda(l)$ we have

$$T(L_{\xi}L_{\eta}^{*}) = \begin{cases} \phi(k+l)L_{\xi}L_{\eta}^{*} & \text{in case 1} \\ \phi(k+l-1)L_{\xi}L_{\eta}^{*} & \text{in case 2.} \end{cases}$$

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B(H) Equivalence $\Phi_{x,y}^{(i)}$

Open questions

For
$$\gamma \in H$$
, define $R_{\gamma} \in B(H)$ as
 $R_{\gamma}(\chi) = \begin{cases} \chi \otimes \gamma & \text{if } i \neq i_1 \\ 0 & \text{if } i = i_1 \end{cases}$

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B(H)Equivalence $\Phi(i)$

Open questions

For
$$\gamma \in H$$
, define $R_{\gamma} \in B(H)$ as

$$R_{\gamma}(\chi) = \begin{cases} \chi \otimes \gamma & \text{if } i \neq i_{1} \\ 0 & \text{if } i = i_{1} \end{cases}$$
For $a = (a_{i}) \in I^{\infty}(\mathbb{N}_{0})$ let

$$D_{a}(\xi_{1} \otimes \cdots \otimes \xi_{n}) = a_{n}\xi_{1} \otimes \cdots \otimes \xi_{n} \text{ and } D_{a}(\Omega) = a_{0}\Omega$$

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B(H)Equivalence $\Phi_{x,v}^{(i)}$

Open questions • For $\gamma \in H$, define $R_{\gamma} \in B(H)$ as $R_{\gamma}(\chi) = \begin{cases} \chi \otimes \gamma & \text{if } i \neq i_1 \\ 0 & \text{if } i = i_1 \end{cases}$ • For $a = (a_i) \in I^{\infty}(\mathbb{N}_0)$ let $D_a(\xi_1 \otimes \cdots \otimes \xi_n) = a_n \xi_1 \otimes \cdots \otimes \xi_n$ and $D_a(\Omega) = a_0 \Omega$ • $\rho(a) = \sum_{\gamma \in \Lambda(1)} R_{\gamma} a R_{\gamma}^*$

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Open questions

For $\gamma \in H$, define $R_{\gamma} \in B(H)$ as
$R_{\gamma}(\chi) = \left\{egin{array}{ccc} \chi\otimes\gamma & ext{if }i eq i_1\ 0 & ext{if }i=i_1 \end{array} ight.$
• For $a=(a_i)\in I^\infty(\mathbb{N}_0)$ let
$D_{a}(\xi_1\otimes\cdots\otimes\xi_n)=a_n\xi_1\otimes\cdots\otimes\xi_n$ and $D_{a}(\Omega)=a_0\Omega$
• $ ho(a) = \sum_{\gamma \in \Lambda(1)} R_{\gamma} a R_{\gamma}^*$
• $\epsilon(a) = \sum_{i \in I} q_i a q_i$ for q_i the projection on

span{
$$\xi \in \Lambda(n) | n \ge 1, \xi = \gamma_1 \otimes \cdots \otimes \gamma_n, \gamma_n \in \mathring{\Gamma}_i$$
}

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Equivalence $\Phi_{X,Y}^{(i)}$

Open questions

For
$$x, y \in l^2(\mathbb{N}_0)$$
 and $a \in B(H)$ put
• $\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x} \rho^n(a) D_{S^n y}^*$

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Radial Multipliers on Reduced Free Products

For $x, y \in l^2(\mathbb{N}_0)$ and $a \in B(H)$ put

• $\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x} \rho^n(a) D_{S^n y}^*$ • $\Phi_{x,y}^{(2)}(a) =$ $\sum_{n=0}^{\infty} D_{(S^*)^n x} a D^*_{(S^*)^n y} + \sum_{n=1}^{\infty} D_{S^n x} \rho^{n-1}(\epsilon(a)) D^*_{S^n y}$

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Open questions

For
$$x, y \in l^{2}(\mathbb{N}_{0})$$
 and $a \in B(H)$ put

$$\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^{\infty} D_{(S^{*})^{n_{x}}} a D_{(S^{*})^{n_{y}}}^{*} + \sum_{n=1}^{\infty} D_{S^{n_{x}}} \rho^{n}(a) D_{S^{n_{y}}}^{*}$$

$$\Phi_{x,y}^{(2)}(a) = \sum_{n=0}^{\infty} D_{(S^{*})^{n_{x}}} a D_{(S^{*})^{n_{y}}}^{*} + \sum_{n=1}^{\infty} D_{S^{n_{x}}} \rho^{n-1}(\epsilon(a)) D_{S^{n_{y}}}^{*}$$

Lemma

For $\xi \in \Lambda(k), \eta \in \Lambda(I)$ we have

$$\rho^{n}(L_{\xi}L_{\eta}^{*}) = L_{\xi}L_{\eta}^{*}P_{\{\zeta \in H \mid |\zeta| \le l+n\}}$$

$$\epsilon(L_{\xi}L_{\eta}^{*}) = \begin{cases} \rho(L_{\xi}L_{\eta}^{*}) & \text{in case } 1\\ L_{\xi}L_{\eta}^{*} & \text{in case } 2 \end{cases}$$

Properties of $\Phi_{x,y}^{(i)}$

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Open questions

Lemma

If
$$\xi \in \Lambda(k), \eta \in \Lambda(I)$$
 then

$$\Phi_{x,y}^{(1)}(L_{\xi}L_{\eta}^{*}) = \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}\right) L_{\xi}L_{\eta}^{*}$$

and

$$\Phi_{x,y}^{(2)}(L_{\xi}L_{\eta}^{*}) = \begin{cases} \sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}L_{\xi}L_{\eta}^{*} & \text{case } 1\\ \sum_{t=0}^{\infty} x(k+t-1)\overline{y(l+t-1)}L_{\xi}L_{\eta}^{*} & \text{case } 2 \end{cases}$$

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Properties of $\phi \in \mathcal{C}$

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Open questions

Lemma

With

$$\phi(n) = \psi_1(n) + \psi_2(n) + c$$

$$\psi_1(k+l) = \sum_{i=1}^{\infty} \sum_{t=0}^{\infty} x_i(k+t) \overline{y_i(l+t)}$$
(1)
$$\psi_2(k+l) = \sum_{i=1}^{\infty} \sum_{t=0}^{\infty} z_i(k+t) \overline{w_i(l+t)}.$$

for $h = \sum_{i=1}^{\infty} x_i \odot y_i$ and $k = \sum_{i=1}^{\infty} z_i \odot w_i$

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B(H) Equivalence $\Phi_{x,y}^{(i)}$

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•
$$T_1 = \sum_{i=1}^{\infty} \Phi_{x_i, y_i}^{(1)}$$
 for $h = \sum_{i=1}^{\infty} x_i \odot y_i$

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B(H) Equivalence $\Phi_{X,Y}^{(i)}$

Open questions

Define

•
$$T_1 = \sum_{i=1}^{\infty} \Phi_{x_i, y_i}^{(1)}$$
 for $h = \sum_{i=1}^{\infty} x_i \odot y_i$
• $T_2 = \sum_{i=1}^{\infty} \Phi_{z_i, w_i}^{(2)}$ for $k = \sum_{i=1}^{\infty} z_i \odot w_i$

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Equivalence $\Phi_{X,Y}^{(i)}$

Open questions

Define

•
$$T_1 = \sum_{i=1}^{\infty} \Phi_{x_i, y_i}^{(1)}$$
 for $h = \sum_{i=1}^{\infty} x_i \odot y_i$

•
$$T_2 = \sum_{i=1}^{\infty} \Phi_{z_i, w_i}^{(2)}$$
 for $k = \sum_{i=1}^{\infty} z_i \odot w_i$

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$$T = T_1 + T_2 + cI$$

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Reduction to B(H)Equivalence $\Phi_{x,y}^{(i)}$

Open questions

Define

•
$$T_1 = \sum_{i=1}^{\infty} \Phi_{x_i, y_i}^{(1)}$$
 for $h = \sum_{i=1}^{\infty} x_i \odot y_i$

•
$$T_2 = \sum_{i=1}^{\infty} \Phi_{z_i, w_i}^{(2)}$$
 for $k = \sum_{i=1}^{\infty} z_i \odot w_i$

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$$T = T_1 + T_2 + cI$$

• \cdots T has the right behavior

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B(H) Equivalence $\Phi_{Y}^{(i)}$

Open questions $\|\Phi_{x_i,y_i}^{(1)}\|_{cb} \le \|x_i\|_2 \|y_i\|_2$

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B(H)Equivalence $\Phi_{x,v}^{(i)}$

Open questions

$$\| \Phi_{x_i, y_i}^{(1)} \|_{cb} \le \| x_i \|_2 \| y_i \|_2$$
$$\| \Phi_{z_i, w_i}^{(2)} \|_{cb} \le \| z_i \|_2 \| w_i \|_2$$

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- $\| \Phi_{x_i,y_i}^{(1)} \|_{cb} \le \|x_i\|_2 \|y_i\|_2 \\ \| \Phi_{z_i,w_i}^{(2)} \|_{cb} \le \|z_i\|_2 \|w_i\|_2$
- $\|T_1\|_{cb} \le \sum_{i=1}^{\infty} \|\Phi_{x_i,y_i}^{(1)}\|_{cb} \le \|h\|_1$

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- $\|\Phi_{x_i,y_i}^{(1)}\|_{cb} \le \|x_i\|_2 \|y_i\|_2$ $\|\Phi_{z_i,w_i}^{(2)}\|_{cb} \le \|z_i\|_2 \|w_i\|_2$
- $\| T_1 \|_{cb} \le \sum_{i=1}^{\infty} \| \Phi_{x_i, y_i}^{(1)} \|_{cb} \le \| h \|_1$
- $\|T_2\|_{cb} \le \sum_{i=1}^{\infty} \|\Phi_{z_i,w_i}^{(2)}\|_{cb} \le \|k\|_1$

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- $\|\Phi_{x_i,y_i}^{(1)}\|_{cb} \le \|x_i\|_2 \|y_i\|_2$ $\|\Phi_{z_i,w_i}^{(2)}\|_{cb} \le \|z_i\|_2 \|w_i\|_2$
- $\|T_1\|_{cb} \leq \sum_{i=1}^{\infty} \|\Phi_{x_i,y_i}^{(1)}\|_{cb} \leq \|h\|_1$
- $||T_2||_{cb} \le \sum_{i=1}^{\infty} \|\Phi_{z_i,w_i}^{(2)}\|_{cb} \le \|k\|_1$
- $\| T \|_{cb} \le \| T_1 \|_{cb} + \| T_2 \|_{cb} + \| cld \|_{cb} \le \| \phi \|_{\mathcal{C}}$

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von Neumann algebra version

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Theorem (Haagerup-M 2010)

Let $(\mathcal{M}, \omega) = \overline{*}_{i \in I}(\mathcal{M}_i, \omega_i)$ be the w*-reduced free product of von Neumann algebras $(\mathcal{M}_i)_{i \in I}$ with respect to normal states $(\omega_i)_{i \in I}$ for which the GNS-representation π_{ω_i} is faithful for all $i \in I$.

If $\phi \in \mathcal{C}$, then there is an unique linear completely bounded normal map

 $M_{\phi}:\mathcal{M}
ightarrow\mathcal{M}$

such that $M_{\phi}(1) = \phi(0)1$ and

$$\mathcal{M}_{\phi}(\mathsf{a}_1\mathsf{a}_2\ldots\mathsf{a}_n)=\phi(n)\mathsf{a}_1\mathsf{a}_2\ldots\mathsf{a}_n$$

whenever $a_j \in \mathring{\mathcal{M}}_{i_j} = ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. Moreover $||\mathcal{M}_{\phi}||_{cb} \leq ||\phi||_{\mathcal{C}}$.

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 $\Phi_{x,y}^{(i)}$

Open questions For which (A_i, ω_i)_{i∈I} holds ||M_φ||_{cb} = ||φ||_C for all φ ∈ C?
[Wysoczanski 1995] True if A_i = C^{*}_r(G_i), |G_i| = ∞, |I| = ∞

Open questions

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- For which $(A_i, \omega_i)_{i \in I}$ holds $||M_{\phi}||_{cb} = ||\phi||_{\mathcal{C}}$ for all $\phi \in \mathcal{C}$?
 - [Wysoczanski 1995] True if $A_i = C_r^*(G_i)$, $|G_i| = \infty$, $|I| = \infty$
- Use (φ_k)_k ⊂ C with finite support, pointwise converging to 1 on CCAP

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- For which $(A_i, \omega_i)_{i \in I}$ holds $||M_{\phi}||_{cb} = ||\phi||_{\mathcal{C}}$ for all $\phi \in \mathcal{C}$?
 - [Wysoczanski 1995] True if $A_i = C_r^*(G_i)$, $|G_i| = \infty$, $|I| = \infty$
- Use (φ_k)_k ⊂ C with finite support, pointwise converging to 1 on CCAP

- Almagamated counterpart
 - \blacksquare scalar valued ϕ
 - $\blacksquare B\text{-valued }\phi$