Amplified Graph Algebras

Adam P. W. Sørensen
Joint work with
Søren Eilers and Efren Ruiz
Graphs

**Definition**

A graph $G$ is a 4-tuple $(G^0, G^1, r, s)$, where $G^0$ is a countable set of vertices, $G^1$ is a countable set of edges, and $r, s : G^1 \to G^0$ are the range and source maps.

<table>
<thead>
<tr>
<th>$G^0$</th>
<th>$G^1$</th>
<th>$r$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bullet$</td>
<td>${e, f}$</td>
<td>$\bullet$</td>
<td>$\bullet$</td>
</tr>
<tr>
<td>$\bullet$</td>
<td>${e_1, e_2, \ldots}$</td>
<td>$\bullet$</td>
<td>$\circ$</td>
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Example

$$e \rightarrow \bullet \rightarrow f$$

$G^0 = \{ \bullet \}$

$G^1 = \{ e, f \}$

$s(e) = \bullet, r(e) = \bullet$

$s(f) = \bullet, r(f) = \bullet$
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$G^0 = \{\bullet, \circ\}$

$G^1 = \{e_1, e_2, \ldots\}$

$s(e_1) = \bullet, r(e_1) = \circ$

$s(e_2) = \bullet, r(e_2) = \circ$
Graph $C^*$-Algebras

**Definition**

Given a graph $G$, we let $C^*(G)$ denote the universal $C^*$-algebra generated by pairwise orthogonal projections $\{p_u \mid u \in G^0\}$ and partial isometries $\{s_e \mid e \in G^1\}$ subject to the relations

- **CK0** $s_e^*s_f = 0$, if $e \neq f$.
- **CK1** $s_e^*s_e = p_{r(e)}$.
- **CK2** $s_es_e^* \leq p_{s(e)}$.
- **CK3**

$$p_u = \sum_{\{e \in G^1 \mid s(e) = u\}} s_es_e^*,$$

if $0 < |s^{-1}(u)| < \infty$. 

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Examples of Graph $C^*$-Algebras

- The complex numbers.
- The Cuntz algebra $\mathcal{O}_n$, $2 \leq n \leq \infty$.
- The unitization of the compact operators.
- The Kirchberg algebra with $K_0 = \mathbb{Z}^2$ and $K_1 = 0$. 
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Classification and the Graphs

**Question**

Are graph algebras classified by K-theory?

**Question**

What does it say about two graphs $E$ and $G$ that $C^*(E)$ is (stably) isomorphic to $C^*(G)$?

**Question**

Is there a (finite) list of “moves” on graphs that generate the relation $G \sim E$ if and only if $C^*(G)$ is stably isomorphic to $C^*(E)$?
Classification and the Graphs

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Graph Operations

Definition

Given a graph $G$, we let the transitive closure of $G$ be the graph $tG$. It has the same vertex set as $G$ and if there is a path from $u$ to $v$ in $G$, then there is an edge in $tG$ with source $u$ and range $v$.

Definition

Given a graph $G$, we define the amplification of $G$ to be the graph $\overline{G}$ with the same vertex set as $G$, but with the property that if there is an edge from $u$ to $v$ in $G$, then there are infinitely many edges from $u$ to $v$ in $\overline{G}$. 
Graph Operations - Pictures

\[
\begin{array}{ccc}
G & \quad tG & \quad \overline{tG} \\
\end{array}
\]
Graph Operations - Pictures
Graph Operations - Pictures

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\[ \begin{array}{c}
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(\infty) \\
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Graph Operations - Pictures

\[ G \quad tG \quad \overline{tG} \]

Diagram:

- Three graphs labeled $G$, $tG$, and $\overline{tG}$ are shown with various nodes and edges.
- The nodes are represented as dots, and the edges are shown as arrows.
- The diagram illustrates different graph operations or transformations.
Graph Operations - Pictures

\[ G \quad tG \quad \overline{tG} \]
The Main Result

Theorem

Let $G$ and $E$ be finite graphs. The following are equivalent:

(i) $C^*(\overline{G}) \simeq C^*(\overline{E})$.

(ii) $C^*(\overline{G})$ and $C^*(\overline{E})$ have the same filtered $K$-theory.

(iii) $C^*(t\overline{G})$ and $C^*(t\overline{E})$ have the same filtered $K$-theory.

(iv) $t\overline{G} \simeq t\overline{E}$.

(v) $C^*(t\overline{G}) \simeq C^*(t\overline{E})$.

Lemma A

$C^*(\overline{G}) \simeq C^*(t\overline{G})$.

Lemma B

$FK(C^*(t\overline{G})) \simeq FK(C^*(t\overline{E})) \implies t\overline{G} \simeq t\overline{E}$. 

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Is It Interesting?

- **Cons:**
  - Specialized graphs.
  - Very boring $K$-theory ($K_0(C^*(G)) = \mathbb{Z}^{|G^0|}$, $K_1(C^*(G)) = 0$).

- **Pros:**
  - $C^*(G)$ can have any (finite) ideal structure.
  - Graphical classification.
  - Nice generalizations (graphs where all vertices are singular).
  - Permanence results (if it looks like an amplified graph algebra, and it quacks like an amplified graph algebra, then it must be an amplified graph algebra).
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**Theorem**

Let $G$ be a graph, $u \in G^0$ an infinite emitter, and $v$ a vertex that $u$ emits infinitely to. Fix an edge $f \in s_G^{-1}(v)$. Let $E$ be the graph with vertex set $G^0$, edge set

$$E^1 = G^1 \cup \{f^n | n \in \mathbb{N}\},$$

and range and source maps that extend those of $G$ and have $r(f^n) = r(f)$ and $s(f^n) = u$. Then $C^*(E) \cong C^*(F)$.

**Example**

![Diagram](image-url)
Theorem

Let $G$ be a graph, $u \in G^0$ an infinite emitter, and $v$ a vertex that $u$ emits infinitely to. Fix an edge $f \in s_{\overline{G}}^{-1}(v)$. Let $E$ be the graph with vertex set $G^0$, edge set

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Move (T)

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Example

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Proof of Lemma A

Lemma

Let $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ be a path in a graph $G$. Let $E$ be the graph with vertex set $G^0$, edge set

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and obvious range and source maps. If there are infinitely many edges parallel to $\alpha_1$ then $C^*(G) \cong C^*(E)$.

Lemma A

If $G$ is a finite graph then $C^*(\overline{G}) \cong C^*(\overline{tG})$. 
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If $G$ is a finite graph then $C^*(\overline{G}) \cong C^*(\overline{tG})$. 
A Note On Lemma B

**Lemma B**

\[ \text{FK}(C^*(\overline{tG})) \cong \text{FK}(C^*(\overline{tE})) \implies \overline{tG} \cong \overline{tE}. \]

**Sketch of Proof**

- For two vertices, \( u \) and \( v \), write \( u \geq v \) if there is a path from \( u \) to \( v \) or \( u = v \).
- For amplified graphs, the relation \( \geq \) is encoded in the ideal structure.
- Use \( \text{Prim}(C^*(\overline{tG})) \cong \text{Prim}(C^*(\overline{tE})) \) to find a bijection \( \psi: \overline{tG}^0 \rightarrow \overline{tE}^0 \) such that \( u \geq v \iff \psi(u) \geq \psi(v) \).
- We are done if we can show that a vertex supports a simple loop in \( \overline{tG} \) if and only if it does in \( \overline{tE} \).
- The ordered \( K_0 \)-group is used to tell us this.
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\[ FK(C^*(tG)) \cong FK(C^*(tE)) \implies tG \cong tE. \]

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Classification From the Outside

**Definition**

Let $C$ be the class of separable, nuclear, simple, purely infinite $C^*$-algebras $\mathfrak{A}$ satisfying the UCT, and with $K_1(\mathfrak{A}) = 0$ and $K_0(\mathfrak{A})$ free.

**Definition**

Let $C_{\text{free}}$ be the class of $C^*$-algebras $\mathfrak{A}$ such that Prim($\mathfrak{A}$) is finite, and for every simple sub-quotient $\mathfrak{B}$ of $\mathfrak{A}$ we have

- $\mathfrak{B}$ is unital or stable, and in $C$ or stably isomorphic to $\mathcal{K}$, and,
- if $\mathfrak{B}$ is unital, then there exists an isomorphism $K_0(\mathfrak{B}) \cong \bigoplus_n \mathbb{Z}$ such that $[1_{\mathfrak{B}}]$ is sent to $(1, \lambda)$.

**Theorem**

The elements of $C_{\text{free}}$ are classified by filtered $K$-theory.
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**Theorem**

*The elements of $\mathcal{C}_{\text{free}}$ are classified by filtered $K$-theory.*
The Payoff

**Proposition**

Let $G$ be a graph with finitely many vertices, no breaking vertices, and with the property that every vertex in $G$ is either an infinite emitter or a sink. We have $C^*(G) \in C_{\text{free}}$.

**Theorem**

Let $\mathcal{A}$ be a unital $C^*$-algebra in $C_{\text{free}}$ with $K_0(\mathcal{A})$ finitely generated. There exists a finite graph $G$ such that $\mathcal{A} \cong C^*(G)$.

**Theorem**

Let $G_1$ and $G_2$ be finite graphs. If $\mathcal{A}$ is a unital $C^*$-algebra and $\mathcal{A}$ fits into the following exact sequence

$$0 \to C^*(G_1) \otimes K \to \mathcal{A} \to C^*(G_2) \to 0$$

then $\mathcal{A} \in C_{\text{free}}$. Consequently, $\mathcal{A} \cong C^*(G)$ for some finite graph $G$. 
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Let $G$ be a graph with finitely many vertices, no breaking vertices, and with the property that every vertex in $G$ is either an infinite emitter or a sink. We have $\mathcal{C}^*(G) \in \mathcal{C}_{\text{free}}$.

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This Is the End