Surgery theory & Group actions III

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Surgery summary

$$L_{n+1}(\mathbb{Z}\pi) \to \mathcal{S}(M) \to \mathcal{N}(M) \to L_n(\mathbb{Z}\pi).$$

$M^n = S^n$.

$\mathcal{S}(S^n) = \Theta_n$ group of smooth homotopy n-spheres.

$\mathcal{N}(S^n) = \pi_n(G/O)$

$$G = \operatorname{colim}_k G_k = \operatorname{colim}_k \{S^{k-1} \to S^{k-1}\}_{\simeq}$$

$$\pi_n(G(k)) = \pi_{n+k-1}(S^{k-1}) \qquad \pi_n G = \pi_n^S.$$

$$\begin{array}{ccc} O(k) & \to & G(k) \\ \downarrow & & \downarrow \\ O(k+1) & \to & G(k+1) \\ \downarrow & & \downarrow \\ O & \subset & G \end{array}$$

Kervaire-Milnor

$$\begin{array}{c} L_{n+1}(\mathbb{Z}) \to \mathcal{S}(S^n) = \Theta_n \text{ group of smooth homotopy n-spheres} \\ \downarrow \\ \mathcal{N}(S^n) = \pi_n(G/O) \end{array}$$

$$bP_{n+1} = \{\Sigma = \partial W^{n+1}\}$$

parallelizable.

G acting freely and smoothly on $S^n \times S^m$?

① $m \gg n$ examples

② $G = A_4$ does not act freely on $S^n \times S^n$.

(equidimensional case).

Which finite gps can act freely and smoothly on some $S^n \times S^n$?

G = extra special p-group $\begin{cases} \text{order } p^3 \\ \exp p \end{cases}$

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}. \quad * \in \mathbb{F}_p.$$

Thm (H.–Ünlü) G_p does act freely + smoothly on $S^{2p-1} \times S^{2p-1}$.

X^{2n} – finitely dominated PD space.
- reducible ν_X
- vanishing surgery obstruction.

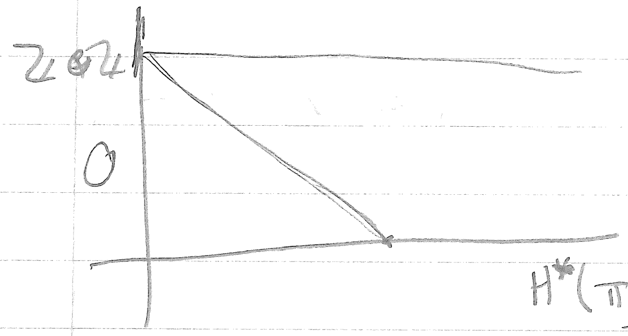
$$\begin{array}{ccc} K(\mathbb{Z}, n) & \longrightarrow & E \\ & & \downarrow \\ & & K(\pi, 1). \end{array}$$

What does a possible quotient of $S^n \times S^n / \mathbb{Z}_p \times \mathbb{Z}_p$ look like.

$\pi = \mathbb{Z}/p \times \mathbb{Z}/p$. n odd.

For $\left(\begin{matrix} K(\mathbb{Z}\oplus\mathbb{Z}, n) \to E \\ \downarrow \\ K(\pi,1) \end{matrix} \right) \rightsquigarrow \Omega_{2n}(E, \nu_E)$.

bordism group.



$dz_1, dz_2 \in H^{n+1}(\pi;\mathbb{Z})$.

$\begin{matrix} \nu_E \\ \downarrow \\ E \end{matrix}$ vector bundle. Lashof-Stong.

$$\Omega_{2n}(E,\nu_E) = \left\{ \begin{matrix} \nu_M & \xrightarrow{b} & \nu_E \\ \downarrow & & \downarrow \\ M^{2n} & \xrightarrow{f} & E \end{matrix} \right\} \Big/ \text{bordism in domain.}$$

closed manifold.

Kreck modified surgery.

G. Lewis (1968) & Gene. $H^*(G_p;\mathbb{Z})$

$dz_1, dz_2 \in H^{n+1}(G_p;\mathbb{Z})$ n odd.

$\dim n = 2p^r-1$.

minimum case $r=1$.

$$1 \to \mathbb{Z}/p \to G_p \to \mathbb{Z}/p \times \mathbb{Z}/p \to 1.$$

$\downarrow$ faithful $\mathbb{C}^\times$

$\alpha, \beta \in H^2$.

induce rep V. $\dim V = p$ $c_p(V) = dz_1$

$dz_2 = \alpha^p + \alpha^{p-1}\beta + \beta^p$

$\nu_E = \{$ Vector bundle $/K(\pi,1)$ pull-back to $E\}$.

$$\begin{array}{ccccc} \Omega_{2n}(E,\nu_E) & \longrightarrow & H_{2n}(E;\mathbb{Z}) & \longrightarrow & H_{2n}(\pi;\mathbb{Z}) \\ \psi & & & & \rotatebox{90}{$\in$} \\ \exists\, [M,f] & \longmapsto & & & \sigma_G \end{array}$$

target class

I. Leary $H^*(G_p;\mathbb{Z})$

$$1 \to S^1 \to \Gamma_p \times \mathbb{Z}/p \times \mathbb{Z}/p \to 1.$$

$$[x,y] = z = e^{2\pi i/p}.$$

$$S^1 \to BG_p \to B\Gamma_p.$$

E. Stein (1977) $D_p \times D_p$ can act freely and smoothly on some $S^n \times S^n$.

$D_p = \langle a, b \mid a^p = b^2 = 1,\ bab^{-1} = a^{-1} \rangle$. p odd prime

<u>Thm</u>$^{(H)}$ $\pi = G_1 \times G_2$ where G_1 and G_2 are finite periodic groups, then π can act freely + smoothly on some $S^n \times S^n$.

Swan: $X(G_1)$, $X(G_2)$ Swan complexes (for each generator $\in H^{n+1}(G;\mathbb{Z})$)

$\tilde{X}(G_1) \simeq S^n$.

$X = X(G_1) \times X(G_2)$ PD-space $\tilde{X} = S^n \times S^n$ $\pi_1 X = \pi$.

① (MTW) choose $X(G_i)$ s.t.

- f.t. PD complex
- reducible
- Gysin are normally cobordant to manifolds whenever $H_i \subseteq G_i$ acts freely, orthogonally on some S^n.

$$\begin{array}{ccc} \nu_{M_1} \times \nu_{M_2} & \xrightarrow{b} & \xi_1 \times \xi_2 \\ \downarrow & & \downarrow \\ M_1 \times M_2 & \xrightarrow{f} & X(G_1) \times X(G_2) \end{array}$$

② $L_*(\mathbb{Z}\pi)$ detecting by restriction to 2-hyperelementary + p-elementary subgps

a) $H < G_1 \times G_2$ hyperelementary.

b) $H_1 \times H_2 < G_1 \times G_2$ $H_i < G_i$ hyperelementary.

Improve Dress induction ⇝ (b).

(B) $\mathcal{G} : D(G) \longrightarrow$ abelian Green.

subgps of G.

$\mathcal{G}(H)$ i.e. Mackey Ind Res.

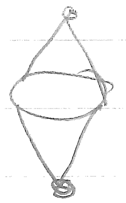
③ Ranicki product formulas.

Kulkarni (1981) Which finite groups can act pseudofreely on S^n?

Actions on S^{2k}

1) π acts freely on S^{2k-1} linearly.
2) $G = D_p$
3) $G =$ tetrahedral, octa, icos $\to (k=1)$

acts on S^2



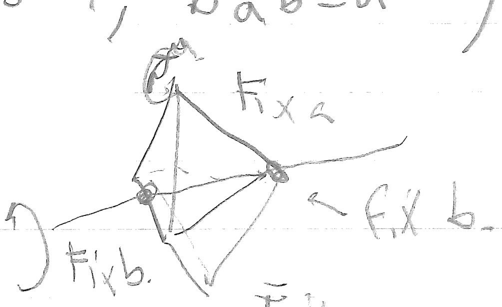
Can $\pi = D_p$ act on S^{2k} for $k>1$?

Answer: no

- k even A. Edmonds
- k odd ~~[illegible]~~ H.

$$G = \langle a, b \mid a^p = b^2 = 1,\ bab = a^{-1} \rangle.$$

$H = \langle a \rangle$.
$K = \langle b \rangle$.



$$C_*(G/H) = C_*(X^H).$$

$$\begin{array}{ccccc} & C_1 & \longrightarrow & C_0 & \\ & \| & & \| & \\ \mathbb{Z}[G/\,]^3 \longrightarrow & \mathbb{Z}[G/?]^3 & \longrightarrow & \mathbb{Z}[G/H^?] \oplus \mathbb{Z}[G/K^?] & \end{array}$$

$\mathbb{Z}[G/H^?](G/V) = \mathbb{Z}\,\mathrm{Map}_G(G/V, G/H)$.

$$\mathbb{Z}[G]^2 \to \mathbb{Z}(G) \to \underset{rk\,2}{\mathbb{Z}[G/H]} \oplus \underset{rk\,3.}{\mathbb{Z}[G/K]^2} \to \mathbb{Z} \to 0.$$

$G = D_p$ $\quad X$ finite G-CW-cpx $\quad \tilde{X} = S^{2k}$

$$D_*(X^?) = (\underline{\mathbb{Z}} \to D_{2k} \to D_{2k-1} \to \cdots \to D_2 \to D_1 \to D_0 \to \mathbb{Z})$$

$$\begin{array}{cccc} D_2 & D_1 & D_0 & \\ \| & \| & \| & \\ C_2 \to & C_1 \to & C_0 \to & \mathbb{Z} \end{array}$$

eval at $G/1$

periodic free resolution.

$$0 \to \mathbb{Z} \to D_{2k}(1) \to \cdots \to D_0(1) \to \mathbb{Z} \to 0.$$

k even length $\equiv 2\ (4)$ ✗ Contradiction

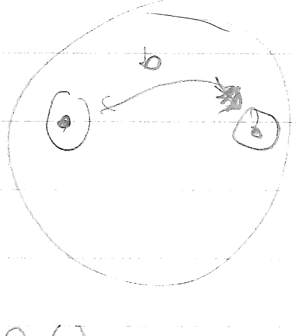
since $d(D_p) = 4$.

k odd ~~G~~ $= D_p$ acting pseudofreely on S^{2k}

$\mathrm{Sing}(S^{2k}, G) \equiv \mathrm{Sing}(S^2, G)$

Equivariant obstruction theory. $\quad S^{2k}$ $\quad k$ odd > 1.

identity on singular set.



$(S^{2k} - S^2, G)$ free $\quad O(2)$-rep.

sing set $\subset S^2$

$S^{2k-3} \times \mathbb{R}^3$.

H-Pedersen (1991) Controlled surgery.

So get
$$\begin{array}{c} S^{2k-3} \times \mathbb{R}^3 \\ \downarrow \\ \mathbb{R}^3 \end{array}$$
Controlled ...

But s.t. this is contradiction (?)