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p-Local compact groups and unstable Adam operations

Def: A p-local finite group is a triple $\mathcal{G}=(S,\mathcal{F},\mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system on a finite p-group S and $\mathcal{L}$ is a linking system associated to $\mathcal{F}$.

p-local compact grous $\xleftrightarrow{\text{are to}}$ compact Lie group
what
p-local finite group $\xleftrightarrow{\text{are to}}$ finite groups.

Examples:

* G compact Lie group, $BG_p^\wedge$
* (X, BX) p-compact group BX p-complete, $H^*(X, \mathbb{F}_p)$ finite
* Γ linear torsion group $\Gamma \leq GL_n(R)$, R ring, Γ torsion group, $B\Gamma_p^\wedge$

To generalize finite to compact:

Replace S by S discrete p-toral

$$\mathbb{Z}/p^\infty = \bigcup_{n\geq 1} \mathbb{Z}/p^n \qquad T \simeq (\mathbb{Z}/p^\infty)^r \text{ discrete torus of rank } r$$

A discrete p-toral group is a group with a normal discrete torus of p-power index.

$$\underbrace{P_0}_{\text{discrete torus.}} \longrightarrow \underbrace{P}_{\text{discrete p-toral}} \longrightarrow \Gamma_P := P/P_0 \text{ finite.}$$

If P is a discret p-toral group then the maximal discrete torus P_0 is called the "connected component" and $\Gamma_P = P/P_0$ is the group of components

Define the <u>order</u> of a discrete p-toral gp
$$|P| = (rk(P_0), |\Gamma_P|).$$

Define a fusion system $\mathcal{F}$ over a discrete p-toral group P in analog way to the finite case.
For saturation: analog to finite + Axiom III – continuity axiom.
- Centric subgroups
- Centric linking system

} analog to finite case.

<u>Obstruction theory</u>:
Existence $\longleftrightarrow H^3(\mathcal{O}(\mathcal{F}^c); \underline{\mathbb{Z}})$
Uniqueness $\longleftrightarrow H^2(\mathcal{O}(\mathcal{F}^c); \underline{\mathbb{Z}})$
<u>Exercice</u>: generalize Chermak's thm to p-local ~~(finite)~~ compact groups.

<u>Metathm</u>: Much of what is true in the case of p-local finite groups is true in the case of p-local compact groups.

The classifying space for $\mathcal{G}(S,\mathcal{F},\mathcal{L})$ is $B\mathcal{G} = |\mathcal{L}|^\wedge_p$.
p-compact groups have rational cohomology.
$$H^*_{\mathbb{Q}_p}(-) = H^*(-, \mathbb{Z}_p) \otimes \mathbb{Q}$$

This is ~~not~~ the same as the rational cohomology.

$$\widehat{H}^*(BP, \mathbb{Q}) = 0 \text{ but } \widetilde{H}^*_{\mathbb{Q}_p}(BP) \neq 0$$

$$\widetilde{H}^*_{\mathbb{Q}_p}(BP) = H^*_{\mathbb{Q}_p}(BP_0)^{\Gamma_P}$$

Thm [BLO] If $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ p-local compact group. Set $W = \mathrm{Aut}_{\mathcal{F}}(S_0)$. Then.

$$H^*_{\mathbb{Q}_p}(B\mathcal{G}) = H^*_{\mathbb{Q}_p}(BS_0)^W$$

the same as for compact Lie groups and for p-local finite groups.

Attemps to approximate p-local compact groups by p-local finite groups.

Unstable Adam operations

* G a compact Lie group.
Map $\psi: BG \to BG$.
s. th $\psi^*: H^*(BG, \mathbb{Q}) \circlearrowleft$
ψ^* is mult by q^i in dimension $2i$
($q \in \mathbb{Z}$ is the degree of the operation).

* p-compact groups (by analogy)
$\psi: BX \circlearrowleft$ such that
$\psi^*: H^*_{\mathbb{Q}_p}(BX) \circlearrowleft$ is multiplication by q^i in dimension $2i$.

Thm (Andersen-Grodal-Møller-Viruel + combinations)
For any p-compact group (X, BX) and any

$\zeta \in \mathbb{Z}_p^\times$ $\exists$ an unstable Adam operation of degree ζ on BX, unique up to homotopy. $\square$

from now on: joint work with Junal, Libman.

<u>Ex</u>

$$\begin{tikzcd} BG(\mathbb{F}_p)^\wedge_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BG^\wedge_p \arrow[d] \\ BG^\wedge_p \arrow[r, "1 \times \psi_q"] & (BG \times BG)^\wedge_p \end{tikzcd}$$

G compact Lie (Friedlander, Quillen, Mislin)

$$\begin{tikzcd} BX(\zeta)^\wedge \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & BX \arrow[d] \\ BX \arrow[r, "1 \times \psi_\zeta"] & BX \times BX \end{tikzcd}$$

p-local finite (Broto-Møller)

<u>Def</u>: S discrete p-toral group. An Adam ~~operation~~ automorphism on S, $\psi \in \mathrm{Aut}(S)$ $\psi|_{S_0}$ is mult. by $\zeta \in \mathbb{Z}_p^\times$ and ψ induces the identity on Γ_p

An unstable Adam operation on $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is a pair $(\psi, \overline{\psi})$ s.th.

1) ψ is a fusion preserving Adam aut of S
2) $\overline{\psi} : \mathcal{L} \to \mathcal{L}$ is an equivalence of categories

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\overline{\psi}"] \arrow[d] & \mathcal{L} \arrow[d] \\ \mathcal{F}^c \arrow[r, "\psi_*"] & \mathcal{F}^c \end{tikzcd}$$

3) $\psi(\hat{g}) = \widehat{\overline{\psi}(g)}$

Geometric def $\psi : B\mathcal{G} \longrightarrow B\mathcal{G}$ such that

$$\begin{tikzcd}
BS \arrow[r] \arrow[dr, bend right, "B\left(\text{Adam's aut}\right)"'] & B\mathcal{G} \arrow[r, "\psi"] & B\mathcal{G} \\
 & BS \arrow[ur] &
\end{tikzcd}$$

$$\mathrm{Ad}(\zeta) \longrightarrow \pi_0(\mathrm{Aut}(B\mathcal{G}))$$

<u>Thm (Junod)</u> For every p-local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ there exists $m \in \mathbb{Z}_{>0}$ such that $\forall\, \zeta \in \mathbb{Z}_p^*$, $\nu_p(\zeta - 1) \geq m$ there exists an unstable Adam operation of degree ζ of $\mathcal{G}$.