Exercises on centric obstruction theory

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1 Definitions

Exercise 1. Let $G$ be a finite group and $S \in \text{Syl}_p(G)$. Recall that $O_S(G)$ is the category with objects the subgroups of $S$ and with morphisms $P_S(G)(P, Q) := Q \backslash N_G(P, Q)$. Show that $O_S(G)$ is equivalent to the category of transitive $G$-sets whose stabilizers are $p$-groups and $G$-maps.

Recall the following from lecture:

Theorem 1.1 (Jackowski-McClure). If $M$ is a $\mathbb{Z}_{(p)}$-module and $F : O_S(G)^{\text{op}} \to \mathbb{Z}_{(p)}\text{-mod}$ is the functor $P \mapsto \mathbb{Z}_{(p)}P$, then

$$H^n(O_S(G); F) = 0$$

for all $n \geq 1$.

Definition 1.2. For $\Gamma$ a finite group, $\Sigma \in \text{Syl}_p(\Gamma)$, and $M$ a $\mathbb{Z}_{(p)}\Gamma$-module, let $F_M : O_{\Sigma}(\Gamma)^{\text{op}} \to \mathbb{Z}_{(p)}\text{-mod}$ be the functor that vanishes off the trivial subgroup and sends $\{1\}$ to $M$. Set

$$\Lambda^*(\Gamma; M) := H^*(O_{\Sigma}(\Gamma); F_M).$$

Theorem 1.3 (Jackowski-McClure-Oliver). If $O$ is either $O_S(G)$ or $O(F^c)$ and $F : O^{\text{op}} \to \mathbb{Z}_{(p)}\text{-mod}$ is a functor that vanishes off of the isomorphism class of $P \leq S$, then for all $n \geq 0$,

$$H^n(O; F) \cong \Lambda^n(\text{Aut}_O(P); F(P)).$$

2 Basic properties

Exercise 2. Show that if $p \nmid |\Gamma|$, then

$$\Lambda^n(\Gamma; M) \cong \begin{cases} M^\Gamma & n = 0 \\ 0 & n \geq 0 \end{cases}.$$

Hint: Does this formula look familiar?

Exercise 3. Show that if $p || |\Gamma|$, then $\Lambda^0(\Gamma; M) = 0$.

Exercise 4. Show that if $|\Sigma| = p^n$, then $\Lambda^n(\Gamma; M) = 0$ for all $m > n$.

Hint: Use Jackowski-McClure. Also induction.

Exercise 5. Show that if $|\Sigma| = p$, then $\Lambda^1(\Gamma; M) \cong M_{N_G(\Sigma)}^\Gamma$.

Exercise 6. Let $G = A_6$, $F = F_S(G)$, and $Z_G : O(F^c) \to \mathbb{Z}_{(p)}\text{-mod}$ be the obstruction functor $P \mapsto Z(P)$. Compute the higher limits of $Z_G$. 

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