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Block algebras, fusion and cohomology.

k algebraically closed field of char p
G finite group, kG group algebra.
B - block of kG (i.e. an indecomposable direct factor of kG).

B - block

<u>invariants of B as an algebra</u>

- B is symmetric (i.e. $B \cong \mathrm{Hom}_k(B, k)$ as B-B bimodule)
- $\mathrm{mod}(B)$ f.g. modules
- $D^b(\mathrm{mod}(B))$ bounded derived category.
- $\ell(B) = \#$ simple B-modules up to isomorphism.
- $HH(B) = \mathrm{Ext}^*_{B \otimes B^{op}}(B, B)$ Hochschild algebra

P <u>defect group</u> of B (p-subgroup of G, unique up to G-conjugation)

- $\mathcal{F} = \mathcal{F}_P(B)$ <u>fusion system</u> of B on P
- $H^*(B) = \lim_{\mathcal{F}} (H^*(-; k))$

  $\cong H^*(|\mathcal{L}|^\wedge_p; k)$

  where $\mathcal{L}$ is a centric linking system
- B determine classes $\alpha_Q \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q); k^\times)$, $Q \in \mathcal{F}$

One understands very little the connection between the two sides above.

- Does $\mathrm{mod}(B)$ determine P, $\mathcal{F}$
- <u>Donovan's conjecture</u>: For fixed P, $\mathcal{F}$ there

should be only finitely many mod(B) from blocks B with P as a defect group and $\mathcal{F}$ the associated fusion system.

- numerical conjectures; e.g. Alperin's weight conjecture predicts $\ell(B)$ in terms of local information.

Fusion systems of blocks:

Q a p-subgroup of G : $kC_G(Q) \hookrightarrow (kG)^Q$

We have a morphism:

$$Br_Q : (kG)^Q \longrightarrow kC_G(Q)$$

induced by the linear projection sending $x \in C_G(Q)$ to itself and $y \in G \setminus C_G(Q)$ to 0.

This is an algebra homomorphism (exercise), called the Brauer homomorphism.

Br_Q sends $Z(kG)$ to $Z(kC_G(Q))$ by sending b to either 0 or a sum of block idempotents.

- P defect group of $B (= kGb)$ is a maximal p-group such that $Br_P(b) \neq 0$.

- (Q, e) is a b-Brauer pair if Q is a p-subgroup and e a block of $kC_G(Q)$ such that
$$Br_Q(b)e = e \quad (\neq 0)$$

Theorem (Alperin - Broué)

The G-set of b-Brauer pairs has a partial order $\leq$, such that

(a) (Q, e), (R, f) Brauer pairs with $R \trianglelefteq Q$ then $Br_Q(f) = e$.

(b) (Q,e) b-Brauer pair, $R \leq Q$, then $\exists!\, f$ s.th $(R,f) \leq (Q,f)$

(c) $(1,b)$ is the unique minimal b-Brauer pair

(d) All maximal b-Brauer pairs are G-conjugate and of the form (P,e) with P a defect group of b.

<u>Fusion system of B on P</u>: fix maximal Brauer pair (P,e). For $Q \leq P$ there is a unique e_Q s.th $(Q,e_Q) \leq (P,e)$.

$\mathcal{F} = \mathcal{F}_P(B) = \mathcal{F}_{(P,e)}(B)$

$\mathrm{Ob}(\mathcal{F}) = \{Q \mid Q \leq P\}$

$\mathrm{Hom}_{\mathcal{F}}(Q,R) = \{\varphi : Q \to R \mid \exists x \in G \;\; \varphi(u) = xux^{-1} \;\; \forall u \in Q \text{ and } {}^x(Q,e_Q) \leq (R,e_R)\}$

$\mathcal{F}_P(B)$ is a subcategory of $\mathcal{F}_S(G)$, where $S \in Syl_p(G)$, $P \leq S$.

The principal block B_0 has $\mathcal{F}_S(G)$ as fusion system.

<u>Thm</u> $\mathcal{F} = \mathcal{F}_P(B)$ is a saturated fusion system

<u>Prop</u> $Q \leq P$, then $Q \in \mathcal{F}^c$

$\iff Z(Q)$ is a defect group of $kC_G(Q)e_Q$.

$\iff kC_G(Q)/Z(Q)\,\overline{e_Q}$ is a matrix algebra.

<u>Associated</u> α_Q: the group $N_G(Q,e_Q)$ acts on $kC_G(Q)/Z(Q)\,e_Q \rightsquigarrow$ get α_Q for

$\mathrm{Aut}_{\mathcal{F}}(Q) \cong N_G(Q,e_Q)/C_G(Q)$.

<u>Gluing problem</u>: does there exist $\alpha \in H^2(\mathcal{F}^c; k^\times)$ such that $\alpha|_{\mathrm{Aut}_{\mathcal{F}}(Q)} = \alpha_Q$ for all $Q \in \mathcal{F}^c$?

Equivalently, is the canonical map.
$H^2(\mathcal{F}^c; k^\times) \longrightarrow \lim_{\mathcal{F}^c} (Q \longrightarrow H^2(\mathrm{Aut}_{\mathcal{F}}(Q), k^\times))$ surjective?

<u>Caution</u> this map is not alway an injection (examples by Sejong Park).

<u>Special case</u> P is abelian.
Then $\mathcal{F} = \mathcal{F}_P(P \rtimes E)$, E a p'-subgroup of $\mathrm{Aut}(P)$; $\mathcal{F}^c = \{P\}$, $\alpha = \alpha_P$ and the gluing problem is trivial.

<u>Broué's abelian defect conjecture</u>: there is an equivalence of derived categories
$D^b(\mathrm{mod}(B)) \cong D^b(\mathrm{mod}(k_\alpha \mathcal{L}))$.

<u>Definition of $k_\alpha \mathcal{C}$</u>, $\mathcal{C}$ a category.
$k_\alpha \mathcal{C}$ has a k-basis the set $\mathrm{Mor}(\mathcal{C})$
$$\psi\varphi = \begin{cases} \alpha(\psi, \varphi)\, \psi\circ\varphi, & \text{if } \psi\circ\varphi \text{ defined} \\ 0 & \text{otherwise.} \end{cases}$$

Alperin's weight conjecture (AWC, 1987)

Let $\mathcal{F}$ be a saturated fusion system on P, $\alpha \in H^2(\mathcal{F}^c; k^\times) \cong H^2(\mathcal{L}; k^\times) = H^2(\mathcal{O}(\mathcal{F}^c); k^\times)$.

A *weight* of $(\mathcal{F}, \alpha)$ is a pair (Q, V) where $Q \in \mathcal{F}^c$ and V is a projective simple $k_\alpha \mathrm{Out}_{\mathcal{F}}(Q)$-module.
If $\varphi : Q \cong Q'$ is an isomorphism in $\mathcal{F}^c$, then φ induces an algebra isomorphism $\hat{\varphi} : k_\alpha \mathrm{Out}_{\mathcal{F}}(Q) \longrightarrow k_\alpha \mathrm{Out}_{\mathcal{F}}(Q')$. Two weights (Q, V), (Q', V') are isomorphic if there exist $\varphi : Q \cong Q'$ in $\mathcal{F}^c$ such that $V \cong \mathrm{Res}_{\hat{\varphi}}(V')$. Set $w(\mathcal{F}, \alpha) = \#$ of isom. classes of weight of $(\mathcal{F}, \alpha)$.

Let B be a block with associated $P, \mathcal{F}, \alpha$.

$$\boxed{\text{AWC}: \quad \ell(B) = w(\mathcal{F}, \alpha)}$$

Other interpretation:

$$\text{AWC} \iff \dim_k(Z(B)) = \sum_{\substack{u \in P/\mathcal{F}\text{-conj} \\ \langle u \rangle \text{ fully centralized}}} w(C_{\mathcal{F}}(u), \tilde{\alpha})$$

with $\tilde{\alpha}$ induced by $\left\{ \begin{array}{ccc} C_{\mathcal{F}}(u)^c & \longrightarrow & \mathcal{F}^c \\ R & \longmapsto & R\langle u \rangle \end{array} \right.$

A chain $\sigma = Q_0 < Q_1 < \cdots < Q_n$ of subgroups of P determines a block B_σ of $N_G(\sigma) = \bigcap_{i=0}^{n} N_G(Q_i, e_{Q_i})$

<u>Thm</u> (Knör – Robinson, 1989)

$AWC \Leftrightarrow \dim_k(Z(B)) = \sum (-1)^{|\sigma|} \dim_k(Z(B_\sigma))$,

where σ runs over a set of representatives of $\mathcal{F}$-conjugacy classes of chains of non-trivial subgroups of P.

<u>Rem</u>: $Z(B) \cong HH^0(B)$

<u>Thm</u> (Külshammer – Robinson, 2002). Let $i > 0$

Then $\dim_k(HH^i(B)) = \sum_\sigma (-1)^{|\sigma|} \dim_k(HH^i(B_\sigma))$.

<u>Rem</u> if σ consists of $\mathcal{F}$-centric subgroups then $mod(B_\sigma) = mod(k_\alpha Aut_{\mathcal{F}}(\sigma))$.

<u>Hilbert series</u> $h_B(t) = \sum_{i \geq 0} \dim_k(HH^i(B)) t^i \in \mathbb{Z}[[t]]$

$AWC \Leftrightarrow h_B(t) = \sum_\sigma (-1)^{|\sigma|} h_{B_\sigma}(t)$.

<u>Hochschild cohomology</u>: $B, P, \mathcal{F}$

$HH^*(B)$, $H^*(B)$.

<u>Theorem</u> (L, 2009) There is a canonical map $H^*(B) \longrightarrow HH^*(B)$ which induces an isomorphism upon taking quotients by nilpotent ideals.

Defect $d = d(B)$; p^d = order of the defect group of B.

joint with Radha Kessar

<u>Theorem</u>: There exists a function $f: \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ such that for any block B with $d = d(B)$ and $n \geq 0$ we have $\dim_k(HH^n(B)) \leq f(d, n)$.

Indredient: Brauer-Feit for any block B with $d = d(B)$.

Thm There is $g : \mathbb{N}_0 \to \mathbb{N}_0$ such that, for any block B with $d = d(B)$, $HH^*(B)$ is generated by generators and relations in degree $\leq g(d)$

Ingredient: Castelnuovo-Mumford regularity $\mathrm{reg}(I, M)$ Zukta

Benson $\mathrm{reg}(H^*(G; k)) \geq 0$, conjecture $= 0$.

Symonds $\mathrm{reg}(H^*(G; k)) = 0 \Rightarrow \mathrm{reg}(HH^*(B)) \leq 0$.

Theorem: Let $d \geq 0$. There are only finitely many $h_B(t)$ of blocks B with $d(B) = d$.

Ingredients Hilberts-Serre: $h_B(t) = \dfrac{f(t)}{\prod_{i=1}^{k}(1-t)^{d_i}}$ is rational, $f \in \mathbb{Z}[t]$, $d_i > 0$.

d_i bounded by the previous theorem.

Serre: $\deg h_B(t)$ bounded in term of regularity $\Rightarrow$ bound for $\deg(f)$.

Thm For fixed $h(t) \in \mathbb{Z}[[t]]$ there are at most finitely many d such that $\exists B$, $d(B) = d$, $h_B(t) = h(t)$.