

Fusion and bisets

Motivation: BLO define p -local finite group $(S, \mathcal{F}, \mathcal{L})$; classifying space $|L|_p^\wedge$ for \mathcal{F} .

H. Miller suggests: take $BS \xrightarrow{f} X$, where X is a p -completed space (+ finiteness properties) also have transfer $\Sigma^\infty X \xrightarrow{t} \Sigma^\infty BS$ such that $t \circ \Sigma^\infty f \approx \text{id}_{\Sigma^\infty X}$ and

$$\begin{array}{ccc}
 \Sigma^\infty X & \xrightarrow{\Delta} & \Sigma^\infty X \wedge \Sigma^\infty X \\
 \text{(FR)} \quad t \downarrow & \curvearrowright & \searrow t \wedge 1 \\
 \Sigma^\infty BS & \xrightarrow{\Delta} & \Sigma^\infty BS \times \Sigma^\infty BS \xrightarrow{\wedge} \Sigma^\infty BS \wedge \Sigma^\infty X
 \end{array}$$

(this diagram induces Frobenius reciprocity in $H^* := H^*(-; \mathbb{F}_p)$)

If such a t exists, we say that $(X \xrightarrow{f} BS)$ is a transfer retract. Equivalently, we say that X is a transfer retract of BS .

"Conjecture" (Miller)

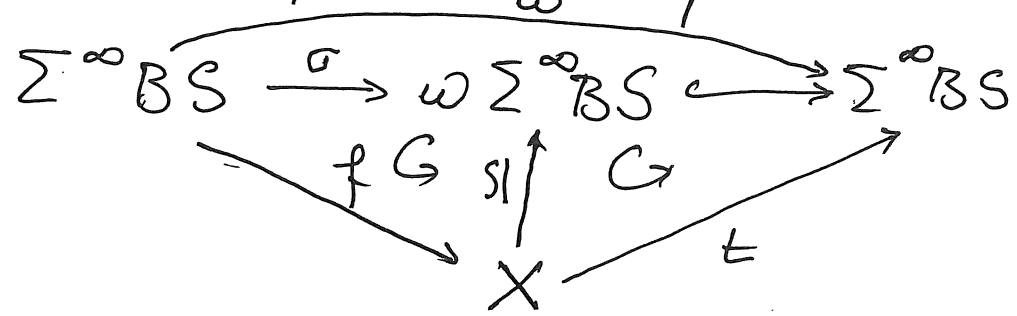
$$\left\{ \begin{array}{l} \text{transfer} \\ \text{retracts} \\ \text{of } S \end{array} \right\} \Big/ \mathcal{N} \quad \longleftrightarrow \quad \left\{ \begin{array}{l} p\text{-local} \\ \text{finite} \\ \text{groups on } S \end{array} \right\} \Big/ \mathcal{N}$$

Thm: " \leftarrow ": For a p -local finite group $(S, \mathcal{F}, \mathcal{L})$, $|L|_p^\wedge$ is a transfer retract

proof (outline)

Have $BS \xrightarrow{\theta} |\mathbb{Z}/p|^\wedge$. How to get the transfer?

For a transfer retract X we have to $\sum_{f=1}^\infty f=1$, so $\omega := \sum_{f=1}^\infty f \circ t$ is idempotent



Exercise: $(FR) \Leftrightarrow \underbrace{(\omega \wedge \omega) \circ \Delta \simeq (\omega \wedge 1) \circ \Delta \circ \omega}_{(FR2)}$

Link to algebra: Segal conj: For S, T finite p -groups.

$$A(S, T)_p^\wedge \xrightarrow{\cong} [\Sigma^\infty BS, \Sigma^\infty BT]$$

\mathbb{Z} -module generated by left-free (S, T) -bisets.

$$T \overset{\text{free}}{\hookrightarrow} X \overset{\text{free}}{\twoheadrightarrow} S$$

In the group case, $\text{tr} \circ \text{incl}$ defined from $[G]$ viewed as an (S, S) -biset

$$\begin{array}{ccc}
 \Sigma^\infty BS & \longrightarrow & \Sigma^\infty BS \\
 \text{incl} \searrow & & \nearrow \text{tr} \\
 & & \Sigma^\infty BG
 \end{array}$$

How to get $[G]$ for \mathbb{F} ?

$$G = \coprod_{s \setminus G / s} S g S = \coprod_{s \setminus G / s} S \times S / \Delta_{s \setminus G / s}$$

Def [Linckelmann - Webb] A characteristic element for \mathcal{F} is $\Omega \in A(S, S)_{(p)}$ such that
 (1) Orbits are $S \times S / \Delta_{\mathcal{F}}$, $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$.

(2) Ω is \mathcal{F} -stable: $\Omega|_{(P, S)} \approx \Omega|_{(P, S)} \in A(P, S)_{(p)}$
 for $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$

(3) $|\Omega/S| \neq 0 \pmod{p}$.

Thm (BLO 2): Every saturated fusion system admits a characteristic biset.

Easy: Every characteristic element satisfies (FR).

Thm (KR, Reeh, Boltje - Danz)

Every saturated fusion system has a unique characteristic idempotent $\omega_{\mathcal{F}} \in A(S, S)_{(p)}$

back to the outline of the proof:

$$\begin{array}{ccccc} \Sigma^{\infty} BS & \xrightarrow{\sigma} & \omega_{\mathcal{F}} \Sigma^{\infty} BS & \xrightarrow{\cong} & \Sigma^{\infty} BS \\ & \searrow \theta & \downarrow h & \leftarrow \text{need equivalence.} & \\ & & \Sigma^{\infty} |\mathcal{L}|_p^{\wedge} & & \end{array}$$

θ factors as $h \circ \sigma$ if and only if $\theta \circ \omega = \theta$.

Thm: A map $\Sigma^{\infty} BS \xrightarrow{h} E$ satisfies $h \circ \omega = h$ if and only if h is \mathcal{F} -stable.

Thm [BLO2]: $BS \xrightarrow{\theta} |L|_p^\wedge$ is \mathcal{F} -stable.

This gives us the map $h: \omega_{\mathcal{F}} \Sigma^\infty BS \rightarrow \Sigma^\infty |L|_p^\wedge$
Is it an \cong ? Everything is $(-)_p^\wedge$ so
it is enough to have an $H^*(-)$ -iso.

Corollary [LW] ω induces an idempotent
 $\omega^*: H^*(BS) \rightarrow H^*(BS)$ with image
 $H^*(\mathcal{F}) := \varprojlim_{\mathcal{F}} H^*(-) \cong \mathcal{F}$ -stable elements
in $H^*(S)$.

[Can replace $H^*(-)$ by any Mackey functor]

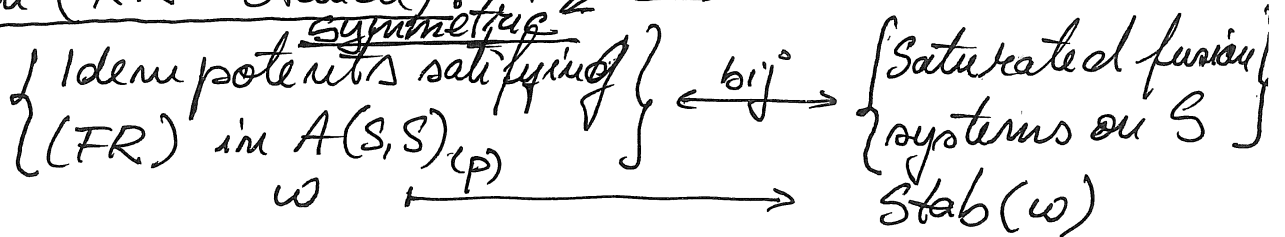
Thm [BLO]: $H^*(|L|_p^\wedge) \cong H^*(\mathcal{F})$

Thus we have $H^*(-)$ iso by thms, so h is \cong .

(end of " \leftarrow ")

For " \rightarrow ": $BS \xrightarrow{\mathcal{F}} X \rightarrow (S, \mathcal{F}_{S,\mathcal{F}}(X), \mathcal{L}_{S,\mathcal{F}}(X))$

Thm (KR-Stancu): $\xleftarrow{\text{Reeh}}$



Left: Show $\mathcal{F}_{S,\mathcal{F}}(X) = \text{Stab}(\omega)$ and
 $\mathcal{L}_{S,\mathcal{F}}(X)$ is the associate linking system.
($X \cong |L_{S,\mathcal{F}}(X)|$).