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Fusion and bisets

Motivation: BLO define p-local finite group $(S, \mathcal{F}, \mathcal{L})$; classifying space $|\mathcal{L}|^\wedge_p$ for $\mathcal{F}$.

H. Miller suggests: take $BS \xrightarrow[\text{mono}]{f} X$, where X is a p-completed space (+ finiteness properties), also have transfer $\Sigma^\infty X \xrightarrow{t} \Sigma^\infty BS$ such that $t \circ \Sigma^\infty f \simeq \mathrm{Id}_{\Sigma^\infty X}$ and

$$
\begin{tikzcd}
\Sigma^\infty X \arrow[r, "\Delta"] \arrow[d, "t"'] \arrow[dr, phantom, "\circlearrowright"] & \Sigma^\infty X \wedge \Sigma^\infty X \arrow[dr, "t\wedge 1"] & \\
\Sigma^\infty BS \arrow[r, "\Delta"] & \Sigma^\infty BS \times \Sigma^\infty BS \arrow[r, "1\wedge t"] & \Sigma^\infty BS \wedge \Sigma^\infty X
\end{tikzcd}
\quad (FR)
$$

(this diagram induces Frobenius reciprocity in $H^* := H^*(-;\mathbb{F}_p)$)

If such a t exists, we say that $(X \xrightarrow{f} BS)$ is a transfer retract. Equivalently, we say that X is a transfer retract of BS.

"Conjecture" (Miller)

$$\left\{\begin{matrix}\text{transfer}\\ \text{retracts}\\ \text{of } S\end{matrix}\right\}\Big/_{\sim} \longleftrightarrow \left\{\begin{matrix}p\text{-local}\\ \text{finite}\\ \text{groups on } S\end{matrix}\right\}\Big/_{\sim}$$

Thm: "$\leftarrow$": For a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, $|\mathcal{L}|^\wedge_p$ is a transfer retract

<u>proof (outline)</u>
Have $BS \xrightarrow{\theta} |\mathcal{L}|^\wedge_p$. How to get the transfer?
For a transfer retract X on S we have $t \circ \Sigma^\infty f = 1$,
so $\omega := \Sigma^\infty f \circ t$ is idempotent

$$\begin{tikzcd}
\Sigma^\infty BS \arrow[r, "\sigma"] \arrow[rr, bend left, "\omega"] \arrow[dr, "f"'] & \omega\Sigma^\infty BS \arrow[r, hook] & \Sigma^\infty BS \\
& X \arrow[u, "\cong"] \arrow[ur, "t"'] &
\end{tikzcd}$$

<u>Exercise</u>: $(FR) \Leftrightarrow \underbrace{(\omega\wedge\omega)\circ\Delta \simeq (\omega\wedge 1)\circ\Delta\circ\omega}_{(FR2)}$

Link to algebra: <u>Segal conj.</u>: For S, T finite p-groups.

$$A(S,T)^\wedge_p \xrightarrow{\cong} [\Sigma^\infty BS, \Sigma^\infty BT]$$

$\mathbb{Z}$-module generated by left-free (S,T)-bisets.
$T \overset{\text{free}}{\curvearrowright} X \curvearrowleft S$

In the group case, $tr \circ incl$ defined from $[G]$ viewed as an (S,S)-biset

$$\begin{tikzcd}
\Sigma^\infty BS \arrow[rr] \arrow[dr, "incl"'] & & \Sigma^\infty BS \\
& \Sigma^\infty BG \arrow[ur, "tr"'] &
\end{tikzcd}$$

How to get $[G]$ for $\mathcal{F}$?
$$G = \coprod_{S\backslash G/S} SgS = \coprod_{S\backslash G/S} S\times S/\Delta^{c_g}_{S\cap {}^gS}$$

<u>Def</u> [Linckelmann-Webb] A characteristic element for $\mathcal{F}$ is $\Omega \in A(S,S)_{(p)}$ such that
(1) Orbits are $S\times S/\Delta_P^{\varphi}$, $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$.

(2) Ω is $\mathcal{F}$-stable: $\Omega|_{(\varphi,S)} \simeq \Omega|_{(P,S)} \in A(P,S)_{(p)}$
for $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$

(3) $|\Omega/S| \not\equiv 0 \pmod p$.

<u>Thm (BLO 2)</u>: Every saturated fusion system admits a characteristic biset.

<u>Easy</u>: ~~Every~~ characteristic element satisfies (FR)

<u>Thm (KR, Reeh, Boltje-Danz)</u>
Every saturated fusion system has a unique charateristic idempotent $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$

back to the outline of the proof:

$$\begin{tikzcd}
\Sigma^\infty BS \arrow[r, "\sigma"] \arrow[rr, bend left, "\omega"] \arrow[dr, "\theta"'] & \omega_{\mathcal{F}}\Sigma^\infty BS \arrow[r, hook] \arrow[d, dashed, "h"', "\leftarrow \text{need equivalence.}"] & \Sigma^\infty BS \\
& \Sigma^\infty |\mathcal{L}|^\wedge_p &
\end{tikzcd}$$

θ factors as $h\circ\sigma$ if and only if $\theta\circ\omega \simeq \theta$.

<u>Thm</u>: A map $\Sigma^\infty BS \xrightarrow{k} E$ satisfies $k\circ\omega = k$ if and only if k is $\mathcal{F}$-stable.

Thm [BLO2] : $BS \xrightarrow{\theta} |\mathcal{L}|_p^\wedge$ is $\mathcal{F}$-stable.

This gives us the map $h : \omega_{\mathcal{F}} \Sigma^\infty BS \to \Sigma^\infty |\mathcal{L}|_p^\wedge$
Is it an $\cong$? Everything is $(-)_p^\wedge$ so it is enough to have an $H^*(-)$-iso.

Corollary [LW] ω induces an idempotent $\omega^* : H^*(BS) \to H^*(BS)$ with image.
$H^*(\mathcal{F}) := \varprojlim_{\mathcal{F}} H^*(-) \simeq \mathcal{F}$-stable elements in $H^*(S)$.
[Can replace $H^*(-)$ by any Mackey functor]

Thm [BLO] : $H^*(|\mathcal{L}|_p^\wedge) \cong H^*(\mathcal{F})$
Thus we have $H^*(-)$ iso by thms, so h is $\simeq$.
(end of "$\leftarrow$")

For "$\longrightarrow$" : $BS \xrightarrow{f} X \longrightarrow (S, \mathcal{F}_{S,f}(X), \mathcal{L}_{S,f}(X))$

Thm (KR-Stancu) : Reeh

$$\left\{ \begin{array}{l} \text{Idempotents satisfying symmetric} \\ \text{(FR) in } A(S,S)_{(p)} \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{l} \text{Saturated fusion} \\ \text{systems on } S \end{array} \right\}$$
$$\omega \longmapsto \mathrm{Stab}(\omega)$$

Left: Show $\mathcal{F}_{S,f}(X) = \mathrm{Stab}(\omega)$ and $\mathcal{L}_{S,f}(X)$ is the associate linking system.
$(X \simeq |\mathcal{L}_{S,f}(X)|)$.