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Classifying (some) reduced fusion systems

$\mathcal{F}_S(G)$, G a finite group, $S \in Syl_p(G)$ is the saturated fusion realized by G on S

Can ou find more examples of exotic fusion systems for $p=2$? $\mathcal{F}$ is exotic if $\mathcal{F}_S(G) \neq \mathcal{F}$, $\forall$ G finite group.

<u>Def</u> A saturated fusion system $\mathcal{F}$ on S is <u>reduced</u> if $O_p(\mathcal{F})=1$, $O^p(\mathcal{F})=\mathcal{F}=O^{p'}(\mathcal{F})$, where $O_p(\mathcal{F})$ = largest $\mathcal{F}$-normal subgroup of S

$O^p(\mathcal{F})$ = unique minimal fusion subsystem of $\mathcal{F}$ of p-power index

$O^{p'}(\mathcal{F})$ = —— " ——

of $\mathcal{F}$ of p'-index.

The uniqness of $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$ proved in BCGLO.

<u>Caution!</u> reduced fusion systems $\neq$ simple fusion system

<u>~~For~~ example:</u> $\mathcal{F}_1, \mathcal{F}_2$ reduced $\Rightarrow \mathcal{F}_1 \times \mathcal{F}_2$ reduced.

Given a saturated fusion system over S

let (1^{st}) $\mathcal{F}_0 = C_{\mathcal{F}}(O_p(\mathcal{F}))$ (or $Z(O_p(\mathcal{F}))$)

(2^{nd}) $\mathcal{F}_0 \supseteq \mathcal{F}_1 = O^p(\mathcal{F}_0) \supseteq \mathcal{F}_2 = O^{p'}(\mathcal{F}_1) \supseteq \cdots$

let $red(\mathcal{F}) = \mathcal{F}_n$ where $O^p(\mathcal{F}_n) = \mathcal{F}_n = O^{p'}(\mathcal{F}_n)$.

<u>Prop</u> $red(\mathcal{F})$ is a reduced saturated fusion system.

<u>Def</u> A saturated fusion system $\mathcal{F}$ is <u>tame</u> if it is realized by a finite group G such that

$$\kappa_G : Out(G) \longrightarrow Out_{typ}(\mathcal{L}_S^c(G)) \cong Out(BG_p^\wedge)$$

is split surjective

<u>Thm A</u> $red(\mathcal{F})$ is tame $=$ $\mathcal{F}$ is tame
(in particular $\mathcal{F}$ is realizable)

<u>Thm B</u> $\mathcal{F}$ not tame $\Rightarrow \exists\ \tilde{\mathcal{F}}$ exotic such that $red(\tilde{\mathcal{F}}) = \mathcal{F}$.

<u>Part II</u> "Classification"

<u>Def</u> $G_0 \lneqq G$ strongly embedded $(p=2)$ if $|G_0|$ is even and $\forall g \in G \setminus G_0$ we have $|gG_0g^{-1} \cap G_0|$ is odd.

- $P \in Ob(\mathcal{F})$ is $\mathcal{F}$-essential if it is $\mathcal{F}$-centric, fully normalized (in $\mathcal{F}$) and $Out_{\mathcal{F}}(P) = Aut_{\mathcal{F}}(P)/Inn\,P$ contains a strongly embedded subgroup, $P \neq S$.

<u>Alperin – Goldschmidt fusion theorem</u>

A saturated fusion system is generated by the automorphisms of $\mathcal{F}$-essential subgroups and those of S.

<u>Def</u> Given S a 2-group, $P \leq S$ is <u>critical</u> if $P \neq S$ and $\exists\, G_0 \leq G$ with
$Out_S(P) \leq G_0 \leq G \leq Out(P)$.
and P is centric in S satifying
$Out_S(P) \in Syl_p(G)$ and G_0 is strongly embedded in G.

Rem: G is candidate for $Out_{\mathcal{F}}(P)$ for some $\mathcal{F}$.

<u>Prop</u> If P is $\mathcal{F}$-essential for some saturated fusion system $\mathcal{F}$, then P is critical.
The converse is not true.

<u>1st step in "classification"</u>
finding all critical subgroups of S.

[Bender] If G contains a strongly embedded subgroup then then $S \in Syl_2(G)$ is cyclic or quaternion-type or ...

- If $\mathcal{F}$ is reduced $\Rightarrow$ $\mathcal{F}$ is non-constrained.
If a saturated fusion system $\mathcal{F}$ contains only one $\mathcal{F}$-essential or <u>one</u> conjugacy class of $\mathcal{F}$-essentials and $Out_{\mathcal{F}}(S)$ is a 2-group. then $\mathcal{F}$ is not reduced.

So one case we're looking for is a saturated fusion system $\mathcal{F}$ over S such that $\exists P$, $\mathcal{F}$-essential, with $|\mathrm{Out}_S(P)| = 2$ and either ($P \ntrianglelefteq S$ and $|N_S(P)/P| = 2$) or ($\exists P_1, P_2$ $\mathcal{F}$-essential, $P_i \trianglelefteq S$ and $|S/P_i| = 2$ $i = 1, 2$)

<u>Amalgams</u>

→ P_i is $\mathcal{F}$-essential $\rightsquigarrow$ $\mathrm{Out}_S(P_i) \in \mathrm{Syl}_2(\mathrm{Out}_{\mathcal{F}}(P_i))$
$\implies \mathrm{Out}_{\mathcal{F}}(P_i) \geq D_{2p_i}$, p_i an odd prime.
Then pick $D_{2p_i} \cong \Gamma_i \leq \mathrm{Out}_{\mathcal{F}}(P_i)$.

→ [BCGLO] $\implies N_{\mathcal{F}}(P_i)$ is realizable by a finite group G_i with $S \in \mathrm{Syl}_2(G_i)$, $P_i \trianglelefteq G_i$ and $G_i/P_i = \Gamma_i = \mathrm{Out}_{G_i}(P_i)$.

→ we have

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, hook] & G_1 \\ G_2 & \end{tikzcd}$$

$\rightsquigarrow G_1 *_S G_2$ infinite
We want finite quotients of $G_1 *_S G_2$

Let $T \leq P_1 \cap P_2$ largest such that $T \trianglelefteq G_i$
Then ~~primitive [illegible]~~ amalgams $G_1/T *_{S/T} G_2/T$ were classified by Fan and Goldschmidt for indices $p_1 = |G_1/S|$ and $p_2 = |G_2/S|$ odd primes.